ONE-PARAMETER PLANAR MOTION ON THE GALILEAN PLANE

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ABSTRACT. Müller [2], on the Euclidean plane \( E^2 \), introduced the one-parameter planar motions and obtained the relation between absolute, relative, sliding velocities and accelerations. Ergin [3] considered the Lorentzian plane \( L^2 \), instead of the Euclidean plane \( E^2 \), and introduced the one-parameter planar motions on the Lorentzian plane and also gave the relations between both the velocities and accelerations.

In this paper, one-parameter motions on the Galilean plane \( G^2 \) are defined. Also the relations between absolute, relative, sliding velocities and accelerations and pole curves are discussed.

1. INTRODUCTION

We consider \( \mathbb{R}^2 \) with the bilinear form

\[
\langle x, y \rangle = x_1 x_2 + \epsilon y_1 y_2
\]

where \( \epsilon \) may be 1, 0 or \(-1\) and \( x = (x_1, y_1), \ y = (x_2, y_2) \in \mathbb{R}^2 \). The distance between two points \( X \) and \( Y \) is defined by

\[
\|x - y\| = |\langle x - y, x - y \rangle|^{\frac{1}{2}}
\]

where \( x \) and \( y \) are the coordinate vectors of the points \( X \) and \( Y \) with respect to the coordinate systems in \( \mathbb{R}^2 \). For \( \epsilon = 1 \) we have the Euclidean plane \( \mathbb{E}^2 \), for \( \epsilon = 0 \) we have the Galilean plane \( \mathbb{G}^2 \), and for \( \epsilon = -1 \) we have the Minkowskian (or Lorentzian) plane \( \mathbb{L}^2 \), (for Lorentzian Plane, see [1]). These are the three Cayley-Klein plane geometries with a parabolic measure of distance. Denote \( \mathbb{R}^2 \) with the bilinear form (1.1) by \( P_\epsilon \), [4].

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Vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{P}_\epsilon$ are orthogonal, written $\mathbf{x} \perp \mathbf{y}$, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Self-orthogonal vectors are called isotropic. For $\epsilon = 1$, only the zero vector is isotropic. For $\epsilon = 0$, zero and vertical vectors are isotropic and, for $\epsilon = -1$, zero vectors and vectors parallel to $(\pm 1, 1)$ are isotropic, [4].

The linear transformation $J : \mathbb{P}_\epsilon \rightarrow \mathbb{P}_\epsilon$ with matrix, also denoted by $J$,

(1.3) $$J = \begin{bmatrix} 0 & -\epsilon \\ 1 & 0 \end{bmatrix}$$

takes any vector $\mathbf{x}$ to an orthogonal vector $J\mathbf{x}$. It is straightforward to check that, if $\mathbf{x}$ is not isotropic and $\mathbf{y}$ is orthogonal to $\mathbf{x}$, then $\mathbf{y} = kJ\mathbf{x}$ for some real number $k$. A circle is the set of points a given distance from a fixed point, the center. The unit circle in $\mathbb{P}_\epsilon$ is the set of points with $\|\mathbf{p}\| = 1$. The unit circles on Euclidean, Galilean and Minkowskian planes are shown in Figure 1, [4].

![Figure 1. The unit circles for $\epsilon = 1, 0, -1$, respectively.](image)

The Galilean unit circle has two branches, the vertical lines $x = \pm 1$, and any point on the $y$–axis is a center. The Minkowskian unit circle has four branches, consisting of a pair of conjugate rectangular hyperbolas with equations $x^2 - y^2 = \pm 1$. Hence, the equation of general unit circle in $\mathbb{P}_\epsilon$ is $x^2 + \epsilon y^2 = \pm 1$. It is not difficult to verify directly from the definition of the matrix exponential as $e^A = \sum A^n / n!$ that

(1.4) $$J = \begin{bmatrix} \cos_\epsilon \phi & -\epsilon \sin_\epsilon \phi \\ \sin_\epsilon \phi & \cos_\epsilon \phi \end{bmatrix}$$

where

(1.5) $$\cos_\epsilon \phi = \sum_{n=0}^{\infty} \frac{(-\epsilon)^n \phi^{2n}}{(2n)!}, \quad \sin_\epsilon \phi = \sum_{n=0}^{\infty} \frac{(-\epsilon)^n \phi^{2n+1}}{(2n + 1)!}.$$ 

For $\epsilon = 1$ these are the usual cosine and sine functions, for $\epsilon = -1$ they are hyperbolic cosine and sine and for $\epsilon = 0$ they are just

(1.6) $$\begin{align*}
\cos_0 \phi &= 1 \\
\sin_0 \phi &= \phi, \quad \text{for all } \phi.
\end{align*}$$

In all case, we obtain

(1.7) $$\cos_\epsilon^2 \phi + \epsilon \sin_\epsilon^2 \phi = 1$$
and

\[(1.8) \quad \partial_\phi \cos \epsilon \phi = -\epsilon \sin \epsilon \phi, \quad \partial_\phi \sin \epsilon \phi = \cos \epsilon \phi.\]

Equating corresponding entries of matrix equation

\[(1.9) \quad e^{J(\phi+\psi)} = e^{J\phi}e^{J\psi}\]

gives the sum formulae

\[(1.10) \quad \cos \epsilon (\phi + \psi) = \cos \epsilon \phi \cos \epsilon \psi - \epsilon \sin \epsilon \phi \sin \epsilon \psi, \]
\[\sin \epsilon (\phi + \psi) = \sin \epsilon \phi \cos \epsilon \psi + \cos \epsilon \phi \sin \epsilon \psi,\]

[4].

1.1. \textbf{Galilean Metric and Galilean Transformation.} The Galilean norm of \(x = (x,y) \in \mathbb{G}^2\) is defined by \(\|x\|_g = \sqrt{\langle x, x \rangle_g} = |x|\). Furthermore, if \(\|x\|_g = 1\), \(x\) is called a \textit{unit vector}, where \(\langle , \rangle_g\) is called the \textit{Galilean inner product} for \(\epsilon = 0\) in the equation (1.1).

On the Galilean Plane, the distance \(d(X,Y)\) between two points \(X = (x_1, y_1)\) and \(Y = (x_2, y_2)\) is defined by the formula

\[(1.11) \quad d(X,Y) = \|YX\|_g = \|x - y\|_g = \sqrt{\langle x - y, x - y \rangle_g} = |x_1 - x_2|\]

and it equals the signed length of the projection \(PQ\) of the segment \(XY\) on the \(x\)-axis (Fig. 2), [5].

![Figure 2. The distance on Galilean plane.](image)

If the distance \(d(X,Y)\) between the points \(X\) and \(Y\) is zero, i.e., \(x_1 = x_2\), then \(X\) and \(Y\) belong to the same special line (parallel to the \(y\)-axis; Fig. 3). For such
points it makes sense to define the *special distance*

\[ (1.12) \quad \delta(X, Y) = |y_1 - y_2|, \]

[5].

**Figure 3.** The special distance on Galilean plane.

Taking \( \varphi \) as the rotation angle between \( x = (x, y) \) and \( x' = (x', y') \) (Fig. 4), we can write

\[
\begin{align*}
x & = r \cos g \theta \\
y & = r \sin g \theta \\
x' & = r \cos (\theta + \varphi) \\
y' & = r \sin (\theta + \varphi),
\end{align*}
\]

where \( \cos g \) and \( \sin g \) are \( \cos_0 \) and \( \sin_0 \) (for \( \epsilon = 0 \), in the equations (1.4-1.10)), respectively.

**Figure 4.** The rotation on Galilean plane.
Then, using the equation (1.10), for \( \epsilon = 0 \), we obtain

\[
\begin{align*}
x' &= x \cos g\varphi + y0 \\
y' &= x \sin g\varphi + y \cos g\varphi.
\end{align*}
\]

From the equation (1.6) (since for all \( \varphi \), \( \cos g\varphi = 1 \) and \( \sin g\varphi = \varphi \)), we get

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
\varphi & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

or

\[
\begin{align*}
x' &= x \\
y' &= \varphi x + y.
\end{align*}
\]

Then, from composed of the transformation (or the shear)

\[
\begin{align*}
x_1 &= x \\
y_1 &= \varphi x + y
\end{align*}
\]

and the transformation (or translation)

\[
\begin{align*}
x' &= x_1 + a \\
y' &= y_1 + b
\end{align*}
\]

we arrive at the formulae

\[
\begin{align*}
x' &= x + a \\
y' &= \varphi x + y + b.
\end{align*}
\]

The equation (1.14) is called Galilean transformation and we remark that the transformation (1.14) map

\begin{enumerate}
\item lines onto lines,
\item parallel lines onto parallel lines,
\item collinear segments onto collinear segments,
\item a figure onto a figure of the same area.
\end{enumerate}

This Galilean transformation belong to the kinematics on \( \mathbb{G}^2 \). Under the Galilean Transformation examining the motion of points of \( \mathbb{G}^2 \) and establishing the invariants are the kinematic geometry of \( \mathbb{G}^2 \). These are called, in other words, the Galilean geometry, [5].

2. KINEMATICS ON THE GALILEAN PLANE

In kinematics, the one-parameter planar motions on the Euclidean plane were given by Müller [2]. Then, the one-parameter planar motions on the Lorentzian plane were given by Ergin [3].
In this section, the one-parameter planar motions on the Galilean plane \( \mathbb{G}^2 \) are defined. Then, the relations between both velocities and accelerations of a point under the one-parameter planar Galilean motions are obtained.

I

Let \( \mathbb{G} \) and \( \mathbb{G}' \) be moving and fixed Galilean planes and \( \{O; g_1, g_2\} \) and \( \{O'; g'_1, g'_2\} \) be their coordinate systems, respectively. By taking

\[
OO' = u = u_1g_1 + u_2g_2, \quad \text{for } u_1, u_2 \in \mathbb{R}
\]

the motion defined by the transformation

\[
x' = x - u
\]

is called a one-parameter planar Galilean motion and denoted by \( B = \mathbb{G}/\mathbb{G}' \), where \( x, x' \) are the coordinate vectors with respect to the moving and fixed rectangular coordinate systems of a point \( X = (x_1, x_2) \in \mathbb{G} \), respectively (Fig. 5). Also the rotation angle \( \varphi \) and the vectors \( x, x' \) and \( u \) are continuously differentiable functions of a time parameter \( t \). Furthermore, at the initial time \( t = 0 \) the coordinate systems coincide. Taking \( \varphi = \varphi(t) \) as the rotation angle between \( g_1 \) and \( g'_1 \) (Fig. 5), we can write

\[
g_1 = g'_1 + \varphi g'_2
\]

\[
g_2 = g'_2.
\]

In this study we assume that

\[
\dot{\varphi}(t) = \frac{d\varphi}{dt} \neq 0,
\]

where "" denotes the derivation with respect to "" and \( \dot{\varphi}(t) \) is called the angular velocity of the motion \( B = \mathbb{G}/\mathbb{G}' \).

Differentiating the equations (2.1) and (2.3) with respect to \( t \), the derivative formulae of the motion \( B = \mathbb{G}/\mathbb{G}' \) are obtained as follows

\[
\dot{g}_1 = \dot{\varphi}g_2
\]

\[
\dot{g}_2 = 0
\]

and

\[
\dot{u} = \dot{u}_1g_1 + (\dot{u}_2 + u_1\dot{\varphi})g_2.
\]

Now, we will define velocities of a point \( X \in \mathbb{G} \) using the derivative formulae of the motion \( B = \mathbb{G}/\mathbb{G}' \):

The velocity of the point \( X \) with respect to \( \mathbb{G} \) is known as the relative velocity \( \mathbf{V}_r \).
and it is defined by $\frac{dx}{dt} = \dot{x}$. Also, for the relative velocity $V_r$, we can write

$$V_r = \dot{x}_1g_1 + \dot{x}_2g_2.$$  

Moreover, if we differentiate the equation (2.2) with respect to $t$, the absolute velocity of the point $X \in \mathbb{G}$ is found as

$$V_a = -\dot{u}_1g_1 + (-\dot{u}_2 - u_1\dot{\phi} + x_1\dot{\phi})g_2 + V_r.$$  

From the equation (2.8), we get the sliding velocity

$$V_f = -\dot{u}_1g_1 + (-\dot{u}_2 - u_1\dot{\phi} + x_1\dot{\phi})g_2.$$  

So we can give the following theorem using the equation (2.7), (2.8) and (2.9).

**Theorem 2.1.** Let $X$ be a moving point on the plane $\mathbb{G}$ and $V_r, V_a$ and $V_f$ be the relative, absolute and sliding velocities of $X$, respectively, under the one-parameter planar motion $B = \mathbb{G}/\mathbb{G}'$. Then

$$V_a = V_f + V_r.$$  

The proof is obvious by using the definitions of velocities above. □

For a general planar motions, there is a point that does not move, which means that its coordinates are the same in both reference coordinate systems $\{O; g_1, g_2\}$ and $\{O'; g'_1, g'_2\}$. This point is called the pole point or the instantaneous rotation pole center, (Fig. 5). In this case, we obtain

$$V_f = 0$$  

or

$$\begin{cases} -\dot{u}_1 = 0 \\ -\dot{u}_2 - u_1\dot{\phi} + x_1\dot{\phi} = 0. \end{cases}$$  

Then for the pole point $P = (p_1, p_2) \in \mathbb{G}$ of the motion $B = \mathbb{G}/\mathbb{G}'$, we have

$$P = \left\{ \begin{array}{l} p_1 = \frac{\alpha(t)}{\dot{\phi}(t)} \\ p_2 = p_2(\lambda(t)) \end{array} \right., \quad \text{for } \lambda \in \mathbb{R}.$$  

**Result 2.1.** Invariant points on both planes at any instant $t$ of $B = \mathbb{G}/\mathbb{G}'$ lie on line parallel to $y-$axis on the plane $\mathbb{G}$. That is, there is only pole line on the plane $\mathbb{G}$ at any instant $t$. For all $t \in I$, this pole lines are parallel to $y-$axis and each other and they constitute bundles of parallel lines.
Using equations (2.9) and (2.11), for the sliding velocity, we can rewrite (2.12)

\[ V_f = \{0g_1 + (x_1 - p_1)g_2\} \dot{\phi}. \]

Now, we can give the following results by the equation (2.12):

**Corollary 2.1.** During the one-parameter plane motion \( B = \mathbb{G}/\mathbb{G}' \), the pole ray \( PX = (x_1 - p_1)g_1 + (x_2 - p_2)g_2 \) and the sliding velocity \( V_f = \{0g_1 + (x_1 - p_1)g_2\} \dot{\phi} \) are perpendicular vectors in the sense of Galilean geometry. That is, \( \langle V_f, PX \rangle_g = 0 \). Then under the motion \( B = \mathbb{G}/\mathbb{G}' \), the focus of the points \( X \in \mathbb{G} \) is an orbit curve that it’s normal pass through the rotation pole \( P \).

**Corollary 2.2.** Under the motion \( B = \mathbb{G}/\mathbb{G}' \), the Galilean norm of the sliding velocity \( V_f \) is

\[ \|V_f\|_g = \|PX\|_g |\dot{\phi}|. \]

That is, during the motion \( B = \mathbb{G}/\mathbb{G}' \), all of the orbits of the points \( X \in \mathbb{G} \) are such curves whose normal lines pass thoroughly the pole \( P \). At any instant \( t \), the motion \( B = \mathbb{G}/\mathbb{G}' \) is a Galilean instantaneous rotation with the angular velocity \( \dot{\phi} \) about the pole point \( P \).

**II**

In this section, we will define relative, absolute, sliding and Coriolis acceleration vectors, during the one-parameter planar motion \( B = \mathbb{G}/\mathbb{G}' \).

Let \( X \) be a moving point of \( \mathbb{G} \). Then the acceleration of the point \( X \) with respect to \( \mathbb{G} \) is known as the relative acceleration and it is defined by \( \frac{d^2x}{dt^2} = \ddot{x} = \dot{V}_r \). Also, for the relative acceleration \( b_r \), we can write

\[ b_r = \ddot{x}_1g_1 + \ddot{x}_2g_2. \]

The acceleration of the point \( X \) with respect to \( \mathbb{G}' \) is known as the absolute acceleration and it is defined by
\begin{equation}
\mathbf{b}_a = \dot{\mathbf{V}}_a = \ddot{x}_1 \mathbf{g}_1 + \{(\dot{x}_1 - \dot{p}_1) \ddot{\phi} - \ddot{p}_1 \dot{\phi} + 2 \dddot{x}_1 \dot{\phi} + \dddot{\phi} \} \mathbf{g}_2
\end{equation}

In the equation (2.14), the expression
\begin{equation}
\mathbf{b}_f = 0 \mathbf{g}_1 + \{(\dot{x}_1 - \dot{p}_1) \ddot{\phi} - \ddot{p}_1 \dot{\phi} \} \mathbf{g}_2
\end{equation}

is called the **sliding acceleration** and

\begin{equation}
\mathbf{b}_c = 0 \mathbf{g}_1 + (2 \dddot{x}_1 \dot{\phi}) \mathbf{g}_2
\end{equation}

is called the **Coriolis acceleration** of the one-parameter planar motion $B = \mathcal{G}/\mathcal{G}'$.

So, we can give the following theorem and corollary using the equations (2.7), (2.14), (2.15) and (2.16):

**Theorem 2.2.** Let $X$ be a moving point on the plane $\mathcal{G}$. Then,

\begin{equation}
\mathbf{b}_a = \mathbf{b}_f + \mathbf{b}_c + \mathbf{b}_r,
\end{equation}

during the one-parameter planar motion $B = \mathcal{G}/\mathcal{G}'$. □

**Corollary 2.3.** During the motion $B = \mathcal{G}/\mathcal{G}'$, the Coriolis acceleration vector $\mathbf{b}_c$ and the relative velocity vector $\mathbf{V}_r$ are perpendicular to each other in the sense of Galilean geometry, i.e. $\langle \mathbf{V}_r, \mathbf{b}_c \rangle_{\mathcal{G}} = 0$.

Under the one-parameter planar motion $B = \mathcal{G}/\mathcal{G}'$, the acceleration pole is characterized by vanishing the sliding acceleration. Therefore, if we take $\mathbf{b}_f = 0$, the acceleration pole point $Q = (q_1, q_2) \in \mathcal{G}$ of the motion $B = \mathcal{G}/\mathcal{G}'$, we get

\begin{equation}
Q = \left\{ \begin{array}{l}
q_1 = p_1 (t) + \dot{p}_1 (t) \frac{\dddot{\phi}(t)}{\dddot{\phi}(t)} \\
q_2 = q_2 (\mu(t))
\end{array} \right., \quad \text{for } \mu \in \mathbb{R}.
\end{equation}

**Result 2.2.** Invariant points on both planes at any instant $t$ of $B = \mathcal{G}/\mathcal{G}'$ lie on line parallel to $y$–axis on the plane $\mathcal{G}$. That is, there is only acceleration pole line on the plane $\mathcal{G}$ at any instant $t$. For all $t \in I$, this acceleration pole lines are parallel to $y$–axis and each other and they constitute bundles of parallel lines.

**References**


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