PROJECTIVE CURVATURE TENSOR OF A SEMI-SYMMETRIC
METRIC CONNECTION IN A KENMOTSU MANIFOLD

AJIT BARMAN AND U. C. DE

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Abstract. The object of the present paper is to study a Kenmotsu manifold
admitting a semi-symmetric metric connection whose projective curvature ten-
sor satisfies certain curvature conditions.

1. Introduction

The product of an almost contact manifold $M$ and the real line $R$ carries a nat-
ural almost complex structure. However if one takes $M$ to be an almost contact
metric manifold and suppose that the product metric $G$ on $M \times R$ is Kaehlerian,
then the structure on $M$ is cosymplectic [12] and not Sasakian. On the other hand
Oubina [15] pointed out that if the conformally related metric $e^{2t}G$, $t$ being the
coordinate on $R$, is Kaehlerian, then $M$ is Sasakian and conversely.

In [19], S. Tanno classified connected almost contact metric manifolds whose au-
tomorphism groups possess the maximum dimension. For such a manifold $M$, the
sectional curvature of plane sections containing $\xi$ is a constant, say $c$. If $c > 0$, $M$
is a homogeneous Sasakian manifold of constant sectional curvature. If $c = 0$, $M$ is
the product of a line or a circle with a Kaehler manifold of constant holomorphic
sectional curvature. If $c < 0$, $M$ is a warped product space $R \times_f C^n$. In 1971,
Kenmotsu studied a class of contact Riemannian manifolds satisfying some special
conditions [14]. We call it Kenmotsu manifold. Kenmotsu manifolds have been
studied by J.B. Jun , U.C. De and G. Pathak [13], C. Özgün and U.C. De [16], U.C.
De and G. Pathak [9], A. Yıldız, U.C. De and B.E. Acet [22] and others.

H.A. Hayden [11] introduced semi-symmetric linear connections on a Riemann-
ian manifold and this was further developed by K. Yano [20], K. Amur and S.S.
Pujar [1], M. Prvanović [17], U.C. De and S.C. Biswas [8], A. Sharfuddin and S.I.
Hussain [18], T.Q. Binh [3], F.Ö. Zengin and S.A. Uysal and S.A. Demirbag [26],
S.K. Chaubey and R.H. Ojha ([6], [7]), H.B. Yılmaz [23] and others.

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Let $M$ be an $n$-dimensional Riemannian manifold of class $C^\infty$ endowed with the Riemannian metric $g$ and $D$ be the Levi-Civita connection on $(M^n, g)$.

A linear connection $\nabla$ defined on $(M^n, g)$ is said to be semi-symmetric [10] if its torsion tensor $T$ is of the form

\begin{equation}
T(X, Y) = \eta(Y)X - \eta(X)Y,
\end{equation}

where $\eta$ is a 1-form and $\xi$ is a vector field given by

\begin{equation}
\eta(X) = g(X, \xi),
\end{equation}

for all vector fields $X \in \chi(M^n)$, $\chi(M^n)$ is the set of all differentiable vector fields on $M^n$.

A semi-symmetric connection $\nabla$ is called a semi-symmetric metric connection [11] if it further satisfies

\begin{equation}
\nabla g = 0.
\end{equation}

A relation between the semi-symmetric metric connection $\nabla$ and the Levi-Civita connection $D$ on $(M^n, g)$ has been obtained by K. Yano [20] which is given by

\begin{equation}
\nabla_X Y = D_X Y + \eta(Y)X - g(X, Y)\xi.
\end{equation}

We also have

\begin{equation}
(\nabla_X \eta)(Y) = (D_X \eta)Y - \eta(X)\eta(Y) + \eta(\xi)g(X, Y).
\end{equation}

Further, a relation between the curvature tensor $R$ of the semi-symmetric metric connection $\nabla$ and the curvature tensor $K$ of the Levi-Civita connection $D$ is given by

\begin{equation}
R(X, Y)Z = K(X, Y)Z + \alpha(X, Z)Y - \alpha(Y, Z)X + g(X, Z)QY - g(Y, Z)QX,
\end{equation}

where $\alpha$ is a tensor field of type $(0,2)$ and $Q$ is a tensor field of type $(1,1)$ which is given by

\begin{equation}
\alpha(Y, Z) = g(QY, Z) = (D_Y \eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}\eta(\xi)g(Y, Z).
\end{equation}

From (1.6) and (1.7), we obtain

\begin{equation}
\check{R}(X, Y, Z, W) = \check{K}(X, Y, Z, W) - \alpha(Y, Z)g(X, W) + \alpha(X, Z)g(Y, W) - g(Y, Z)\alpha(X, W) + g(X, Z)\alpha(Y, W),
\end{equation}

where

\begin{equation}
\check{R}(X, Y, Z, W) = g(R(X, Y)Z, W), \quad \check{K}(X, Y, Z, W) = g(K(X, Y)Z, W).
\end{equation}
The Projective curvature tensor is an important tensor from the differential geometric point of view. Let \( M \) be a \((2n + 1)\)-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of \( M \) and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then \( M \) is said to be locally projectively flat. For \( n \geq 1 \), \( M \) is locally projectively flat if and only if the projective curvature tensor \( P \) vanishes. Here the projective curvature tensor \( P \) with respect to the semi-symmetric metric connection is defined by

\[
P(X, Y)Z = R(X, Y)Z - \frac{1}{2n} [S(Y, Z)X - S(X, Z)Y],
\]

From (1.10), it follows that

\[
\tilde{P}(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) - \frac{1}{2n} [S(Y, Z)g(X, W) - S(X, Z)g(Y, W)],
\]

and

\[
\tilde{P}(X, Y, Z, W) = g(P(X, Y)Z, W),
\]

for \( X, Y, Z, W \in \chi(M) \), where \( S \) is the Ricci tensor with respect to the semi-symmetric metric connection. In fact \( M \) is projectively flat if and only if it is of constant curvature [21]. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

In this paper we study the projective curvature tensor on Kenmotsu manifold with respect to the semi-symmetric metric connection. The paper is organized as follows: After introduction in section 2, we give a brief account of the Kenmotsu manifolds. In section 3, we investigate the quasi-projectively flat Kenmotsu manifolds with respect to the semi-symmetric metric connection and we prove that the manifold is an \( \eta \)-Einstein manifold. Section 4 is devoted to study \( \xi \)-projectively flat Kenmotsu manifolds with respect to the semi-symmetric metric connection. Section 5 deals with \( \phi \)-projectively flat Kenmotsu manifolds with respect to the semi-symmetric metric connection. Finally, we study \( P.S = 0 \) in a Kenmotsu manifold with respect to the semi-symmetric metric connection.

2. Kenmotsu Manifolds

Let \( M \) be an \((2n + 1)\)-dimensional almost contact metric manifold with an almost contact metric structure \((\phi, \xi, \eta, g)\) consisting of a \((1, 1)\) tensor field \( \phi \), a vector field \( \xi \), a 1-form \( \eta \) and a Riemannian metric \( g \) on \( M \) satisfying [4]

\[
\phi^2(X) = -X + \eta(X)\xi, \quad g(X, \xi) = \eta(X),
\]

\[
\eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta(\phi(X)) = 0,
\]
\[(2.3) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\]

for all vector fields \(X, Y\) on \(M\). If an almost contact metric manifold satisfies

\[(2.4) \quad (D_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X,\]

then \(M\) is called a Kenmotsu manifold [14]. From the above relations, it follows that

\[(2.5) \quad D_X \xi = X - \eta(X)\xi,\]

\[(2.6) \quad (D_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y).\]

Moreover the curvature tensor \(K\) and the Ricci tensor \(\tilde{S}\) of the Kenmotsu manifold with respect to the Levi-Civita connection satisfies

\[(2.7) \quad K(X, Y)\xi = \eta(Y)X - \eta(Y)\eta(X),\]

\[(2.8) \quad K(\xi, X)Y = \eta(Y)X - g(X, Y)\xi,\]

\[(2.9) \quad K(X, \xi)Y = g(X, Y)\xi - \eta(Y)X,\]

\[(2.10) \quad \tilde{S}(\phi X, \phi Y) = \tilde{S}(X, Y) + 2\eta(X)\eta(Y),\]

\[(2.11) \quad \tilde{S}(X, \xi) = -2\eta(X).\]

We state the following lemma which will be used in the next section:

**Lemma 2.1.** [14] Let \(M\) be an \(\eta\)-Einstein Kenmotsu manifold of the form \(S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)\). If \(b = \text{constant}\) (or, \(a = \text{constant}\)), then \(M\) is an Einstein one.
3. Quasi-Projectively flat Kenmotsu manifolds with respect to the semi-symmetric metric connection

**Definition 3.1.** A Kenmotsu manifold is said to be quasi-projectively flat with respect to the semi-symmetric metric connection if

\[ g(P(X, Y)Z, \phi W) = 0. \]

**Definition 3.2.** A Kenmotsu manifold is said to be an \( \eta \)-Einstein manifold if its Ricci tensor \( \tilde{S} \) of the Levi-Civita connection is of the form

\[ \tilde{S}(X, Y) = a g(X, Y) + b \eta(X) \eta(Y), \]

where \( a \) and \( b \) are smooth functions on the manifold.

Using (1.7), (2.2) and (2.6) in (1.6), we obtain

\[ R(X, Y)Z = K(X, Y)Z - 3g(Y, Z)X + 3g(X, Z)Y + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y + 2g(Y, Z)\eta(X)\xi - 2g(X, Z)\eta(Y)\eta(W). \]

Using (1.9) in (3.3), we get

\[ \tilde{R}(X, Y, Z, W) = \tilde{K}(X, Y, Z, W) - 3g(Y, Z)g(X, W) + 3g(X, Z)g(Y, W) + 2\eta(Y)\eta(Z)g(X, W) - 2\eta(X)\eta(Z)g(Y, W) + 2g(Y, Z)\eta(X)\eta(W) - 2g(X, Z)\eta(Y)\eta(W). \]

Contracting \( X \) in (3.3), we have

\[ S(Y, Z) = \tilde{S}(Y, Z) - 2(3n - 1)g(Y, Z) + 2(2n - 1)\eta(Y)\eta(Z). \]

Putting \( Z = \xi \) in (3.5) and using (2.11), (2.1) and (2.2), we obtain

\[ S(Y, \xi) = -4n\eta(Y). \]

Again contracting \( Y \) and \( Z \) in (3.5), it follows that

\[ r = \tilde{r} - 2n(6n - 1). \]

where \( r \) and \( \tilde{r} \) are the scalar curvature with respect to the semi-symmetric metric connection and the Levi-Civita connection respectively.

Putting \( X = \phi X \) and \( Y = \phi Y \) in (1.11) and using (1.12), we get

\[ g(P(\phi X, Y)Z, \phi W) = \tilde{R}(\phi X, Y, Z, \phi W) - \frac{1}{2n} [S(Y, Z)g(\phi X, \phi W) - S(\phi X, Z)g(Y, \phi W)]. \]

We begin with the following:
Lemma 3.1. Let $M$ be a $(2n + 1)$-dimensional Kenmotsu manifold. If $M$ satisfies
\[ g(P(\phi X, Y)Z, \phi W) = 0, \quad X, Y, Z, W \in \chi(M), \]
then $M$ is an $\eta$-Einstein manifold.

Proof: Using (3.9) in (3.8), we have
\[ \tilde{R}(\phi X, Y, Z, \phi W) = \frac{1}{2n} [S(Y, Z)g(\phi X, \phi W) - S(\phi X, Z)g(Y, \phi W)]. \]

Again using (3.4) and (3.5) in (3.10), it follows that
\[ \tilde{K}(\phi X, Y, Z, \phi W) = \frac{1}{n} g(Y, Z)g(\phi X, \phi W) - \frac{1}{n} g(\phi X, Z)g(Y, \phi W) - \frac{1}{n} \eta(Y)\eta(Z)g(\phi X, \phi W) + \frac{1}{n} \eta(Y)\eta(Z)g(\phi X, \phi W). \]

Let $\{e_1, ..., e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in $M$, then $\{\phi e_1, ..., \phi e_{2n}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (3.11) and summing over $i = 1$ to $2n$, we get
\[ \sum_{i=1}^{2n} \tilde{K}(\phi e_i, Y, Z, \phi e_i) = \frac{1}{n} \sum_{i=1}^{2n} g(Y, Z)g(\phi e_i, \phi e_i) - \frac{1}{n} \sum_{i=1}^{2n} g(\phi e_i, Z)g(Y, \phi e_i) - \frac{1}{n} \sum_{i=1}^{2n} \eta(Y)\eta(Z)g(\phi e_i, \phi e_i) + \frac{1}{n} \sum_{i=1}^{2n} \eta(Y)\eta(Z)g(\phi e_i, \phi e_i). \]

From (3.12), we obtain
\[ \tilde{S}(Y, Z) = (4n - 2)g(Y, Z) - 4n\eta(Y)\eta(Z). \]

Therefore, \[ \tilde{S}(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z), \]
where $a = 4n - 2$ and $b = -4n$.

This result shows that the manifold is an $\eta$-Einstein manifold. This proves the Lemma.

In view of Lemma (3.1), we can state the following theorem:

Theorem 3.1. If a Kenmotsu manifold is quasi-projectively flat with respect to the semi-symmetric metric connection, then the manifold is an $\eta$-Einstein manifold.

Since $a$ and $b$ are both constant, by Lemma (2.1), we get the following:
Corollary 3.1. If a Kenmotsu manifold is quasi-projectively flat with respect to the semi-symmetric metric connection, then the manifold is an Einstein manifold.

4. $\xi$-Projectively flat and $\phi$-Projectively flat Kenmotsu manifolds with respect to the semi-symmetric metric connection

Let $C$ be the Weyl conformal curvature tensor of a $(2n + 1)$-dimensional manifold $M$. Since at each point $p \in M$ the tangent space $\chi_p(M)$ can be decomposed into the direct sum $\chi_p(M) = \phi(\chi_p(M)) \oplus L(\xi_p)$, where $L(\xi_p)$ is a 1-dimensional linear subspace of $\chi_p(M)$ generated by $\xi_p$. Then we have a map:

$$C : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \longrightarrow \phi(\chi_p(M)) \oplus L(\xi_p).$$

It may be natural to consider the following particular cases:

1. $C : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \longrightarrow L(\xi_p)$, i.e., the projection of the image of $C$ in $\phi(\chi_p(M))$ is zero.

2. $C : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \longrightarrow \phi(\chi_p(M))$, i.e., the projection of the image of $C$ in $L(\xi_p)$ is zero.

3. $C(X, Y) = 0$.

Here the cases 1, 2 and 3 are conformally symmetric, $\xi$-conformally flat and $\phi$-conformally flat respectively. The cases (1) and (2) were considered in [5] and [24] respectively. The case (3) was considered in [25] for the case $M$ is a K-contact manifold. Furthermore in [2], the authors studied contact metric manifolds satisfying (3). Analogous to the definition of $\xi$-conformally flat and $\phi$-conformally flat, we give the following definitions:

Definition 4.1. A Kenmotsu manifold with respect to the semi-symmetric metric connection is said to be $\xi$-projectively flat if

$$P(X, Y)\xi = 0.$$  

Definition 4.2. A Kenmotsu manifold is said to be $\phi$-projectively flat with respect to the semi-symmetric metric connection if

$$g(P(\phi X, \phi Y)\phi Z, \phi W) = 0,$$

where $X, Y, Z, W \in \chi(M)$.

Putting $Z = \xi$ in (3.3) and using (2.1) and (2.2), it follows that

$$R(X, Y)\xi = K(X, Y)\xi + \eta(X)Y - \eta(Y)X.$$
Using (2.7) in (4.5), we obtain

\[(4.6)\]

\[R(X, Y)\xi = 2K(X, Y)\xi.\]

Putting \(Z = \xi\) in (1.10), we have

\[(4.7)\]

\[P(X, Y)\xi = R(X, Y)\xi - \frac{1}{2n}[S(Y, \xi)X - S(X, \xi)Y].\]

Using (3.6) and (4.6) in (4.7), we get

\[(4.8)\]

\[P(X, Y)\xi = 0.\]

Hence we can state the following theorem:

**Theorem 4.1.** If a Kenmotsu manifold admits a semi-symmetric metric connection, then the Kenmotsu manifold is \(\xi\)-Projectively flat with respect to the semi-symmetric metric connection.

Putting \(Y = \phi Y\) and \(Z = \phi Z\) in (3.8), we get

\[(4.9)\]

\[g(P(\phi X, \phi Y)\phi Z, \phi W) = g(R(\phi X, \phi Y)\phi Z, \phi W) - \frac{1}{2n}[S(\phi Y, \phi Z)g(\phi X, \phi W) - S(\phi X, \phi Z)g(\phi Y, \phi W)].\]

Using (2.1), (2.2), (3.3) and (3.5) in (4.9), we have

\[(4.10)\]

\[g(P(\phi X, \phi Y)\phi Z, \phi W) = g(K(\phi X, \phi Y)\phi Z, \phi W) - \frac{1}{2n}[\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi W) - \tilde{S}(\phi X, \phi Z)g(\phi Y, \phi W)] - \frac{1}{n}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)].\]

Again using (4.4) in (4.10), we obtain

\[(4.11)\]

\[g(K(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2n}[\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi W) - \tilde{S}(\phi X, \phi Z)g(\phi Y, \phi W)] + \frac{1}{n}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)].\]

Let \(\{e_1, \ldots, e_{2n}, \xi\}\) be a local orthonormal basis of vector fields in \(M\), then \(\{\phi e_1, \ldots, \phi e_{2n}, \xi\}\) is also a local orthonormal basis. Putting \(X = W = e_i\) in (4.11) and summing over \(i = 1\) to \(2n\), we get

\[(4.12)\]

\[\sum_{i=1}^{2n}g(K(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{1}{2n}\sum_{i=1}^{2n}[\tilde{S}(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - \tilde{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i)] + \frac{1}{n}\sum_{i=1}^{2n}[g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)].\]
From (4.12), it follows that

\[(4.13) \bar{S}(\phi Y, \phi Z) = 2(2n - 1)g(\phi Y, \phi Z)\].

Using (2.3) and (2.10) in (4.13), we obtain

\[(4.14) \bar{S}(Y, Z) = 2(2n - 1)g(Y, Z) - 2(3n - 1)\eta(Y)\eta(Z)\].

Therefore,

\[\bar{S}(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z),\]

where \(a = 2(2n - 1)\) and \(b = -2(3n - 1)\).

We can state the following theorem:

**Theorem 4.2.** If a Kenmotsu manifold is \(\phi\)-projectively flat with respect to the semi-symmetric metric connection, then the manifold is an \(\eta\)-Einstein manifold.

Since \(a\) and \(b\) are both constant, by Lemma (2.1), we get the following:

**Corollary 4.1.** If a Kenmotsu manifold is \(\phi\)-projectively flat with respect to the semi-symmetric metric connection, then the manifold is an Einstein manifold.

5. **Kenmotsu manifolds with respect to the semi-symmetric metric connection satisfying \(P.S = 0\)**

In this section we consider Kenmotsu manifold with respect to the semi-symmetric metric connection \(M^{2n+1}\) satisfying condition

\[(P(U, Y).S)(Z, X) = 0\]

Then we have

\[(5.1) S(P(U, Y)Z, X) + S(Z, P(U, Y)X) = 0.\]

Putting \(U = \xi\) in (5.1), it follows that

\[(5.2) S(P(\xi, Y)Z, X) + S(Z, P(\xi, Y)X) = 0.\]

Putting \(X = \xi\) and using (3.5) and (3.6) in (1.10), we get

\[(5.3) P(\xi, Y)Z = R(\xi, Y)Z - \frac{1}{2n} [\bar{S}(Y, Z)\xi - 2(3n - 1)g(Y, Z)\xi + 2(2n - 1)\eta(Y)\eta(Z)\xi + 4n\eta(Z)Y].\]

Again putting \(X = \xi\) in (3.3) and using (2.8), we obtain

\[(5.4) R(\xi, Y)Z = 2[\eta(Z)Y - g(Y, Z)\xi].\]
Using (3.5), (3.6), (5.3) and (5.4) in (5.2), it follows that

\[(5.5) \tilde{S}(Y, Z) = 2(n-1)g(Y, Z) + 2(1-2n)\eta(Y)\eta(Z).\]

Therefore, \(\tilde{S}(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z),\)

where \(a = 2(n-1)\) and \(b = 2(1-2n)\).

We can state the following theorem:

**Theorem 5.1.** If a Kenmotsu manifold with respect to the semi-symmetric metric connection satisfying \(PS = 0\), then the manifold is an \(\eta\)-Einstein manifold.

Since \(a\) and \(b\) are both constant, by Lemma (2.1), we get the following:

**Corollary 5.1.** If a Kenmotsu manifold with respect to the semi-symmetric metric connection satisfying \(PS = 0\), then the manifold is an Einstein manifold.

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DEPARTMENT OF MATHEMATICS, KABI-NAZRUL MAHAVIDYALAYA, SONAMURA-799181, TRIPURA, INDIA.
E-mail address: ajitbarmanaw@yahoo.in

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALCUTTA, KOLKATA 700019, WEST BENGAL, INDIA.
E-mail address: ucde@yahoo.com