CURVES OF GENERALIZED $AW(k)$-TYPE IN EUCLIDEAN SPACES

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Abstract. In this study, we consider curves of generalized $AW(k)$-type of Euclidean $n$-space. We give curvature conditions of these kind of curves.

1. Introduction

In [7], the first author and A. West defined the notion of submanifolds of $AW(k)$-type. Since then, many works have been done related to these type of manifolds (for example, see [15], [5], [6] and [3]). In [15], the first author and B. Kılıç studied curves and surfaces of $AW(k)$-type. Further, in [27], C. Özgür and F. Gezgin carried out the results for where given in [5] to Bertrand curves and new special curves defined in [13] by S. Izumiya and N. Takeuchi. For example, in [5] and [15], the authors gave curvature conditions and characterizations related to these curves in $\mathbb{R}^n$. Also many results are obtained in Lorentzian spaces in [17], [22], [21], [18] and [8]. In [32], D. Yoon investigate curvature conditions of curves of $AW(k)$-type in Lie group $G$. Recently, C. Özgür and the second author studied some types of slant curves of pseudo-Hermitian $AW(k)$-type in [26].

In the present study, we give a generalization of $AW(k)$-type curves in Euclidean $n$-space $\mathbb{E}^n$. We also give curvature conditions of these type of curves.

2. Basic Notation

Let $\gamma : I \subseteq \mathbb{R} \rightarrow \mathbb{E}^n$ be a unit speed curve in $\mathbb{E}^n$. The curve $\gamma$ is called a Frenet curve of osculating order $d$ if its higher order derivatives $\gamma'(s), \gamma''(s), \ldots, \gamma^{(d)}(s)$ ($d \leq n$) are linearly independent and $\gamma'(s), \gamma''(s), \ldots, \gamma^{(d+1)}(s)$ are no longer linearly independent for all $s \in I$. To each Frenet curve of order $d$, one can associate an orthonormal $d$-frame $v_1, v_2, \ldots, v_d$ along $\gamma$ (such that $\gamma'(s) = v_1$) called the Frenet $d$-frame and $(d-1)$ functions $\kappa_1, \kappa_2, \ldots, \kappa_{d-1} : I \rightarrow \mathbb{R}$ called the Frenet curvatures.

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such that the Frenet formulas are defined in the usual way:

\[
\begin{align*}
D_{v_1} v_1 &= \kappa_1 v_2, \\
D_{v_1} v_2 &= -\kappa_1 v_1 + \kappa_2 v_3, \\
&\vdots \\
D_{v_1} v_i &= -\kappa_{i-1} v_{i-1} + \kappa_i v_{i+1}, \\
D_{v_1} v_d &= -\kappa_{d-1} v_{d-1},
\end{align*}
\]

(2.1)

where \(3 \leq i \leq d-1\).

3. Curves of Generalized AW\((k)\)-type

Let \(\gamma\) be a unit speed curve in \(n\)-dimensional Euclidean space \(\mathbb{E}^n\). By the use of Frenet formulas (2.1), we obtain the higher order derivatives of \(\gamma\) as follows:

\[
\begin{align*}
\gamma'(s) &= \kappa_1 v_2, \\
\gamma''(s) &= -\kappa_1^2 v_1 + \kappa_1' v_2 + \kappa_1 K_2 v_3, \\
\gamma'''(s) &= -3\kappa_1\kappa_1' v_1 + \left(\kappa_1'' - \kappa_1^3 - \kappa_1 K_2'\right) v_2 \\
&\quad + (2\kappa_1' K_2 + \kappa_1 K_2') v_3 + \kappa_1 K_2 K_3 v_4, \\
\gamma^{(4)}(s) &= \left[-3(\kappa_1')^2 - 4\kappa_1\kappa_1'' + \kappa_1 + \kappa_1^2 K_2^2\right] v_1 \\
&\quad + \left(\kappa_1''' - 6\kappa_1\kappa_1' - 3\kappa_1^2 K_2 - 3\kappa_1 K_2 K_2'\right) v_2 \\
&\quad + (3\kappa_1' K_2 + 3\kappa_1^2 K_2 - \kappa_1 K_2^2 + \kappa_1 K_2'' - \kappa_1 K_2 K_3) v_3 \\
&\quad + (3\kappa_1' K_2 K_3 + 2\kappa_1 K_2 K_3 + K_1 K_2 K_3) v_4 + \kappa_1 K_2 K_3 K_4 v_5.
\end{align*}
\]

(3.1)

Let us write

\[
\begin{align*}
N_1 &= \kappa_1 v_2, \\
N_2 &= \kappa_1' v_2 + \kappa_1 K_2 v_3, \\
N_3 &= \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4, \\
N_4 &= \mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4 + \mu_5 v_5,
\end{align*}
\]

(3.2)

where

\[
\begin{align*}
\lambda_2 &= \kappa_1'' - \kappa_1^3 - \kappa_1 K_2', \\
\lambda_3 &= 2\kappa_1' K_2 + \kappa_1 K_2', \\
\lambda_4 &= \kappa_1 K_2 K_3
\end{align*}
\]

(3.3)

and

\[
\begin{align*}
\mu_2 &= \kappa_1''' - 6\kappa_1\kappa_1' - 3\kappa_1^2 K_2 - 3\kappa_1 K_2 K_2', \\
\mu_3 &= 3\kappa_1' K_2 + 3\kappa_1^2 K_2 - \kappa_1 K_2^2 + \kappa_1 K_2'' - \kappa_1 K_2 K_3, \\
\mu_4 &= 3\kappa_1' K_2 K_3 + 2\kappa_1 K_2 K_3 + \kappa_1 K_2 K_3', \\
\mu_5 &= \kappa_1 K_2 K_3 K_4
\end{align*}
\]

(3.4)

are differentiable functions.

We give the following definition:

**Definition 3.1.** Frenet curves are

i) of generalized AW\((1)\)-type if they satisfy \(N_4 = 0\),

ii) of generalized AW\((2)\)-type if they satisfy

\[
\|N_2\|^2 N_4 = (N_2, N_4) N_2,
\]

(3.5)

iii) of generalized AW\((3)\)-type if they satisfy

\[
\|N_1\|^2 N_4 = (N_1, N_4) N_1,
\]

(3.6)

iv) of generalized AW\((4)\)-type if they satisfy

\[
\|N_3\|^2 N_4 = (N_3, N_4) N_3,
\]

(3.7)
v) of generalized AW(5)-type if they satisfy
\[ N_4 = a_1 N_1 + b_1 N_2, \]
vii) of generalized AW(6)-type if they satisfy
\[ N_4 = a_2 N_1 + b_2 N_3, \]
viii) of generalized AW(7)-type if they satisfy
\[ N_4 = a_3 N_2 + b_3 N_3, \]
where \( a_i, b_i \) (1 \( \leq i \leq 3 \)) are non-zero real valued differentiable functions.

**Remark 3.1.** We use notation GAW\((k)\)-type for the curves of generalized AW\((k)\)-type.

Geometrically, a curve of GAW\((k)\)-type is a curve whose fifth derivative’s normal part is either zero or linearly dependent with one or two of its previous derivatives’ normal parts.

Firstly, we give the following proposition:

**Proposition 3.1.** The osculating order of a Frenet curve of any GAW\((k)\)-type can not be bigger than or equal to 5.

**Proof.** Let \( \gamma : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^n \) be a Frenet curve of osculating order \( d \). If \( \gamma \) is of any GAW\((k)\)-type, since none of \( N_i \) (1 \( \leq i \leq 3 \)) contains a component in the direction of \( v_5 \), we find \( \mu_5 = \kappa_1 \kappa_2 \kappa_3 \kappa_4 = 0 \). This concludes \( d \leq 4 \), which completes the proof.

Using equations 3.2 and Definition 3.1, we obtain the following main theorem:

**Theorem 3.1.** Let \( \gamma \) be a unit speed Frenet curve of osculating order \( d \leq 4 \) in \( n \)-dimensional Euclidean space \( \mathbb{E}^n \). Then \( \gamma \) is
i) of GAW(1)-type if and only if
\[ \mu_2 = \mu_3 = \mu_4 = 0, \]
ii) of GAW(2)-type if and only if
\[ \mu_4 = 0, \quad \kappa_1 \kappa_2 \mu_2 - \kappa_4' \mu_3 = 0, \]
iii) of GAW(3)-type if and only if
\[ \mu_3 = \mu_4 = 0, \]
iv) of GAW(4)-type if and only if
\[ \lambda_2 \mu_3 - \lambda_3 \mu_2 = 0, \quad \lambda_2 \mu_4 - \lambda_4 \mu_2 = 0, \]
v) of GAW(5)-type if and only if
\[ \mu_2 = a_1 \kappa_1 + b_1 \kappa_1', \quad \mu_3 = b_1 \kappa_1 \kappa_2, \quad \mu_4 = 0, \]
vi) of GAW(6)-type if and only if
\[ \mu_2 = a_2 \kappa_1 + b_2 \lambda_2. \]
we have

\( N \)

Since \( v \)

\[ (3.11) \]

\[ (3.12) \]

\[ \text{Hence, we can write} \]

\[ \mu_2 = a_3 \kappa'_1 + b_3 \lambda_2, \]

\[ \mu_3 = a_3 \kappa_1 \kappa_2 + b_3 \lambda_3, \]

\[ \mu_4 = b_3 \lambda_4. \]

Proof. i) Let \( \gamma \) be of GAW(1)-type. Then, from equations (3.2) and Definition 3.1, we have \( N_4 = \mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4 = 0 \). Since \( v_2, v_3 \) and \( v_4 \) are linearly independent, we get \( \mu_2 = \mu_3 = \mu_4 = 0 \). The sufficiency is trivial.

ii) Let \( \gamma \) be of GAW(2)-type. If we calculate \( \|N_2\|^2 \) and \( \langle N_2, N_4 \rangle \), by the use of equations (3.2) and (3.5), we obtain

\[ [(\kappa'_1)^2 + \kappa_1^2 \kappa_2^2] (\mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4) = (\kappa'_1 \mu_2 + \kappa_1 \kappa_2 \mu_3)(\kappa'_1 v_2 + \kappa_1 \kappa_2 v_3). \]

Since \( v_2, v_3 \) and \( v_4 \) are linearly independent, we find \( \mu_4 = 0 \) and \( \kappa_1 \kappa_2 \mu_2 - \kappa'_1 \mu_3 = 0 \). Conversely, if \( \mu_4 = 0 \) and \( \kappa_1 \kappa_2 \mu_2 - \kappa'_1 \mu_3 = 0 \), one can easily show that equation (3.5) is satisfied.

iii) Let \( \gamma \) be of GAW(3)-type. We get \( \|N_1\|^2 = \kappa_1^2 \) and \( \langle N_1, N_4 \rangle = \kappa_1 \mu_2 \). So, if we write these equations in (3.6), we have

\[ \kappa_1^2 (\mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4) = \kappa_1 \mu_2 (\kappa'_1 v_2). \]

Thus, \( \mu_3 = \mu_4 = 0 \). Converse theorem is clear.

iv) Let \( \gamma \) be of GAW(4)-type. We can easily calculate \( \|N_3\|^2 = \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \) and \( \langle N_3, N_4 \rangle = \lambda_2 \mu_2 + \lambda_3 \mu_3 + \lambda_4 \mu_4 \). So equation (3.7) gives us

\[ (\lambda_2^2 + \lambda_3^2 + \lambda_4^2)(\mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4) = (\lambda_2 \mu_2 + \lambda_3 \mu_3 + \lambda_4 \mu_4)(\lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4). \]

Hence, we can write

\begin{align*}
(3.11) & \quad (\lambda_2^2 + \lambda_3^2 + \lambda_4^2) \mu_2 = (\lambda_2 \mu_2 + \lambda_3 \mu_3 + \lambda_4 \mu_4) \lambda_2, \\
(3.12) & \quad (\lambda_2^2 + \lambda_3^2 + \lambda_4^2) \mu_3 = (\lambda_2 \mu_2 + \lambda_3 \mu_3 + \lambda_4 \mu_4) \lambda_3, \\
(3.13) & \quad (\lambda_2^2 + \lambda_3^2 + \lambda_4^2) \mu_4 = (\lambda_2 \mu_2 + \lambda_3 \mu_3 + \lambda_4 \mu_4) \lambda_4.
\end{align*}

If we multiply (3.11) with \( \lambda_3 \) and use equation (3.12), we find \( \lambda_2 \mu_3 - \lambda_3 \mu_2 = 0 \). Multiplying (3.11) with \( \lambda_4 \) and using equation (3.13), we have \( \lambda_2 \mu_4 - \lambda_4 \mu_2 = 0 \). Conversely, it is easy to show that equation (3.7) is satisfied if \( \lambda_2 \mu_3 - \lambda_3 \mu_2 = 0 \) and \( \lambda_2 \mu_4 - \lambda_4 \mu_2 = 0 \).

v) Let \( \gamma \) be of GAW(5)-type. Then, in view of equations (3.8) and (3.2), we can write

\[ \mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4 = a_1 (\kappa_1 v_2) + b_1 (\kappa'_1 v_2 + \kappa_1 \kappa_2 v_3), \]

which gives us \( \mu_2 = a_1 \kappa_1 + b_1 \kappa'_1, \mu_3 = b_1 \kappa_1 \kappa_2 \) and \( \mu_4 = 0 \). Conversely, if these last three equations are satisfied, one can show that \( N_4 = a_1 N_1 + b_1 N_2 \).

vi) Let \( \gamma \) be of GAW(6)-type. By definition, we have \( N_4 = a_2 N_1 + b_2 N_3 \), that is,

\[ \mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4 = a_2 (\kappa_1 v_2) + b_2 (\lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4). \]

Since \( v_2, v_3 \) and \( v_4 \) are linearly independent, we can write

\[ \mu_2 = a_2 \kappa_1 + b_2 \lambda_2, \]

\[ \mu_3 = b_2 \lambda_3, \]
\[ \mu_4 = b_2 \lambda_4. \]

Conversely, if these last equations are satisfied, then we easily show that \( N_4 = a_2 N_1 + b_2 N_3. \)

vii) Let \( \gamma \) be of \( GAW(7) \)-type. Then using equations (3.10) and (3.2), we obtain
\[ \mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4 = a_3 (\kappa'_1 v_2 + \kappa_1 \kappa_2 v_3) + b_3 (\lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4). \]

Thus
\[ \mu_2 = a_3 \kappa'_1 + b_3 \lambda_2, \]
\[ \mu_3 = a_3 \kappa_1 \kappa_2 + b_3 \lambda_3, \]
\[ \mu_4 = b_3 \lambda_4. \]

Conversely, let \( \gamma \) be a curve satisfying the last three equations. It is easily found that \( N_4 = a_3 N_2 + b_3 N_3. \) \( \square \)

From now on, we consider Frenet curves whose first curvature \( \kappa_1 \) is a constant. We give curvature conditions of such a curve to be of \( GAW(k) \)-type. We can state following propositions:

**Proposition 3.2.** Let \( \gamma : I \subseteq \mathbb{R} \to \mathbb{E}^n \) be a unit speed Frenet curve of osculating order \( d \leq 4 \) with \( \kappa_1 \) constant. Then \( \gamma \) is of \( GAW(1) \)-type if and only if it is a straight line or a circle.

**Proof.** Let \( \gamma \) be of \( GAW(1) \)-type. Since \( \kappa_1 \) constant, using (3.4) and Theorem 3.1, we find
\[ \mu_2 = -3 \kappa_1 \kappa_2 \kappa'_2 = 0, \] (3.14)
\[ \mu_3 = -\kappa_1^3 \kappa_2 - \kappa_1 \kappa_3^3 + \kappa_1 \kappa'_2 - \kappa_1 \kappa_2 \kappa_3^2 = 0, \] (3.15)
\[ \mu_4 = 2 \kappa_1 \kappa_2 \kappa_3 + \kappa_1 \kappa_2 \kappa'_3 = 0. \] (3.16)

If \( \kappa_1 = 0 \), then \( \gamma \) is a straight line and above three equations are satisfied. Let \( \kappa_1 \) be a non-zero constant. If \( \kappa_2 = 0 \), then \( \gamma \) is a circle and equations (3.14), (3.15) and (3.16) are satisfied again. Assume that \( \kappa_2 \neq 0 \). Then (3.14) gives us \( \kappa'_2 = 0 \), that is, \( \kappa_2 \) is a constant. In this case, from equation (3.15), we get \((\kappa_1^2 + \kappa_2^2 + \kappa_3^2) = 0\), which means \( \kappa_1 = \kappa_2 = \kappa_3 = 0 \). This is a contradiction. So \( \kappa_2 = 0 \).

Conversely, let \( \gamma \) be a straight line or a circle. Thus \( \kappa_1 = 0 \); or \( \kappa_1 \) constant and \( \kappa_2 = 0 \). So \( \mu_2 = \mu_3 = \mu_4 = 0 \), which completes the proof. \( \square \)

**Proposition 3.3.** Let \( \gamma : I \subseteq \mathbb{R} \to \mathbb{E}^n \) be a unit speed Frenet curve of osculating order \( d \leq 4 \) with \( \kappa_1 \) constant. Then \( \gamma \) is of \( GAW(2) \)-type if and only if
i) it is a straight line; or
ii) it is a circle; or
iii) it is a helix of order 3 or 4.

**Proof.** Let \( \gamma \) be of \( GAW(2) \)-type. Since \( \kappa_1 \) constant, using (3.4) and Theorem 3.1, we obtain
\[ \mu_4 = 2 \kappa_1 \kappa'_2 \kappa_3 + \kappa_1 \kappa_2 \kappa'_3 = 0, \] (3.17)
\[ \kappa_1 \kappa_2 (-3 \kappa_1 \kappa_2 \kappa'_2) = 0. \] (3.18)

One can easily see that \( \kappa_2 \) and \( \kappa_3 \) must be constants. Thus, \( \gamma \) can be a straight line, a circle or a helix of order 3 or 4. Conversely, if \( \gamma \) is one of these curves, the proof is clear using Theorem 3.1. \( \square \)
Proposition 3.4. Let \( I \subseteq \mathbb{R} \to \mathbb{E}^n \) be a unit speed Frenet curve of osculating order \( d \leq 4 \) with \( \kappa_1 = \text{constant} \). Then \( \gamma \) is of GAW(3)-type if and only if

i) it is a straight line; or

ii) it is a circle; or

iii) it is a Frenet curve of osculating order 3 satisfying the second order non-linear ODE

\[ \kappa'' = \kappa_2(\kappa_1^2 + \kappa_3^2); \text{ or} \]

and its third curvature satisfies the second order non-linear ODE

\[ (3.19) \quad \kappa''' = \frac{3(\kappa_3')^2}{2\kappa_3} + 2\kappa_3(\kappa_1^2 + \kappa_3^2) + 2c^2 = 0, \]

and where \( c > 0 \) is an arbitrary constant.

Proof. Let \( \gamma \) be of GAW(3)-type. Since \( \kappa_1 = \text{constant} \), using (3.4) and Theorem 3.1, we have

\[ (3.20) \quad \mu_3 = -\kappa_1^3 - \kappa_1 \kappa_3^2 + \kappa_1 \kappa_3'' - \kappa_1 \kappa_3 \kappa_3' = 0, \]

\[ (3.21) \quad \mu_4 = 2\kappa_1 \kappa_2^2 \kappa_3 + \kappa_1 \kappa_2 \kappa_3' = 0. \]

If \( d = 1 \) or \( d = 2 \), we obtain line and circle cases, both of which do not contradict above two equations. Let \( d = 3 \). Then \( \kappa_1 = \text{constant} > 0, \kappa_2 > 0 \) and \( \kappa_3 = 0 \). (3.21) is satisfied directly and (3.20) gives us

\[ \kappa'' = \kappa_2(\kappa_1^2 + \kappa_3^2), \]

which is a second order non-linear ODE. Now, let \( d = 4 \). Thus, \( \kappa_1 = \text{constant} > 0, \kappa_2 > 0 \) and \( \kappa_3 > 0 \). If we solve (3.21), we find

\[ (3.22) \quad \kappa_2 = \frac{c}{\sqrt{\kappa_3}}, \]

where \( c > 0 \) is an arbitrary constant. Then

\[ (3.23) \quad \kappa'' = \frac{-c\kappa_3'}{2\kappa_3^{5/2}}. \]

If we multiply equation (3.20) with \( \frac{\kappa_3'}{\kappa_3} \), using (3.22) and (3.23), we obtain the second order non-linear ODE (3.19). Conversely, if \( \gamma \) is one of these curves, one can show that \( \mu_3 = \mu_4 = 0. \)

Proposition 3.5. Let \( \gamma : I \subseteq \mathbb{R} \to \mathbb{E}^n \) be a unit speed Frenet curve of osculating order \( d \leq 4 \) with \( \kappa_1 = \text{constant} \). Then \( \gamma \) is of GAW(4)-type if and only if

i) it is a straight line; or

ii) it is a circle; or

iii) it is a Frenet curve of osculating order 3 satisfying the second order non-linear ODE

\[ 3\kappa_2(\kappa_1^2 + \kappa_3^2)^2 = (\kappa_2^2 + \kappa_2^3) [\kappa''_2 - \kappa_2(\kappa_1^2 + \kappa_3^2)^2]; \text{ or} \]

\[ (3.24) \quad 3\kappa_2(\kappa_1^2 + \kappa_3^2)^2 = (\kappa_2^2 + \kappa_2^3) [\kappa''_2 - \kappa_2(\kappa_1^2 + \kappa_3^2)^2]; \text{ or} \]
iv) it is a Frenet curve of osculating order 4 with

\[ \kappa^2_2\kappa_3 = c \left( \kappa^2_1 + \kappa^2_2 \right)^{3/2} \]

and its curvatures satisfy

\[ 3\kappa_2(\kappa'_2)^2 = (\kappa^2_1 + \kappa^2_2) \left[ \kappa''_2 - \kappa_2(\kappa^2_1 + \kappa^2_2 + \kappa^3_3) \right]. \]

Here, \( c \) is an arbitrary constant.

Proof. Let \( \gamma \) be of \( \text{GAW}(4) \)-type. Since \( \kappa_1 = \) constant, using (3.3), (3.4) and Theorem 3.1, we find

\[ \begin{align*}
(3.26) \quad & (-\kappa^4_1 - \kappa_1\kappa^3_2)(-\kappa^4_1\kappa_2 - \kappa_1\kappa^3_2 + \kappa_1\kappa''_2 - \kappa_1\kappa_2\kappa^3_3) - (\kappa_1\kappa'_2)(-3\kappa_1\kappa_2\kappa'_2) = 0, \\
(3.27) \quad & (-\kappa^4_1 - \kappa_1\kappa^3_2)(2\kappa_1\kappa'_2\kappa_3 + \kappa_1\kappa_2\kappa'_3) - (\kappa_1\kappa_2\kappa_3)(-3\kappa_1\kappa_2\kappa'_3) = 0.
\end{align*} \]

(3.26) and (3.27) give us

\[ \begin{align*}
(3.28) \quad & 3\kappa_2(\kappa'_2)^2 = (\kappa^2_1 + \kappa^2_2) \left[ \kappa''_2 - \kappa_2(\kappa^2_1 + \kappa^2_2 + \kappa^3_3) \right], \\
(3.29) \quad & (2\kappa^4_1 - \kappa^3_2)\kappa_3\kappa'_2 + \kappa_2(\kappa^2_1 + \kappa^2_2)\kappa'_3 = 0.
\end{align*} \]

Now, if \( \kappa_1 = 0 \), then \( \gamma \) is a straight line and equations (3.26) and (3.27) are satisfied. Let \( \kappa_1 \) be a non-zero constant. If \( \kappa_2 = 0 \), then \( \gamma \) is a circle. Let \( \kappa_2 > 0 \) and \( \kappa_3 = 0 \). Then, from equation (3.28), we obtain (3.24). Now, let \( d = 4 \). Then, using equation (3.29), we can write

\[ \int \frac{(2\kappa^4_1 - \kappa^3_2)}{\kappa_2(\kappa^2_1 + \kappa^2_2)} \, d\kappa_2 + \int \frac{1}{\kappa_3} \, d\kappa_3 = \ln c. \]

Remember that \( \kappa_1 > 0 \) is a constant. So we find

\[ 2\ln(\kappa_2) - \frac{3}{2} \ln(\kappa^2_1 + \kappa^2_2) + \ln(\kappa_3) = \ln c, \]

which gives us (3.25). Furthermore, \( \gamma \) must also satisfy (3.28). Conversely, if \( \gamma \) is one of the curves above, we can show that (3.26) and (3.27) are satisfied. \( \square \)

Proposition 3.6. Let \( \gamma : I \subseteq \mathbb{R} \to \mathbb{E}^n \) be a unit speed Frenet curve of osculating order \( d \leq 4 \) with \( \kappa_1 = \) constant. Then \( \gamma \) is of \( \text{GAW}(5) \)-type if and only if

i) it is a straight line; or

ii) it is a Frenet curve of osculating order 3 with

\[ \kappa_2 \neq \text{constant} \]

and

\[ \kappa''_2 \neq \kappa_2(\kappa^2_1 + \kappa^2_2); \text{ or} \]

iii) it is a Frenet curve of osculating order 4 with

\[ \kappa_2 \neq \text{constant}, \kappa_3 \neq \text{constant}, \]

\[ \kappa''_2 \neq \kappa_2(\kappa^2_1 + \kappa^2_2 + \kappa^3_3) \]

and

\[ \kappa_2 = \frac{c}{\sqrt{\kappa_3}}, \]

where \( c > 0 \) is an arbitrary constant.
Proof. Let $\gamma$ be of $GAW(5)$-type. Since $\kappa_1 = \text{constant}$, by the use of Theorem 3.1 and equations (3.4), we have

(3.30) \[-3\kappa_1 \kappa_2 \kappa_2' = a_1 \kappa_1,\]
(3.31) \[-\kappa_1^3 \kappa_2 - \kappa_1 \kappa_2^3 + \kappa_1 \kappa_2'' - \kappa_1 \kappa_2 \kappa_3^2 = b_1 \kappa_1 \kappa_2,\]
(3.32) \[2 \kappa_1 \kappa_2' \kappa_3 + \kappa_1 \kappa_2 \kappa_3' = 0.\]

If $d = 1$, then $\gamma$ is a straight line and above equations are satisfied. If $d = 2$, then $\gamma$ is a circle. From (3.30), we find $a_1 = 0$, which contradicts the definition. Now, let $d = 3$. Then, using (3.30) and (3.31), we find

\[a_1 = -3 \kappa_2 \kappa_2',\]
\[b_1 = \frac{\kappa_2''}{\kappa_2} - \kappa_1^2 - \kappa_2^2.\]

Since $a_1$ and $b_1$ are non-zero functions, then $\kappa_2 \neq \text{constant}$ and $\kappa_2'' \neq \kappa_2 (\kappa_1^2 + \kappa_2^2)$. Finally, let $d = 4$. Then, equation (3.32) gives us

(3.33) \[\kappa_2 = \frac{c}{\sqrt{\kappa_3}},\]

where $c > 0$ is an arbitrary constant. In this case, from (3.30) and (3.31), we find

(3.34) \[a_1 = -3 \kappa_2 \kappa_2',\]
(3.35) \[b_1 = \frac{\kappa_2''}{\kappa_2} - \kappa_1^2 - \kappa_2^2 - \kappa_3^2.\]

Thus, (3.33) and (3.34) give us

(3.36) \[\kappa_2 \neq \text{constant}, \quad \kappa_3 \neq \text{constant}.\]

Also, from (3.35), we can write

(3.37) \[\kappa_2'' \neq \kappa_2 (\kappa_1^2 + \kappa_2^2 + \kappa_3^2).\]

Converse proposition is trivial. \qed

Proposition 3.7. Let $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^n$ be a unit speed Frenet curve of osculating order $d \leq 4$ with $\kappa_1 = \text{constant}$. Then $\gamma$ is of $GAW(6)$-type if and only if

i) it is a straight line; or

ii) it is a circle; or

iii) it is a Frenet curve of osculating order 3 with $\kappa_2 \neq \text{constant}$,

$\kappa_2'' \neq \kappa_2 (\kappa_1^2 + \kappa_2^2)$

and

$\kappa_2'' \neq \kappa_2 (\kappa_1^2 + \kappa_2^2) + \frac{3 \kappa_2 (\kappa_2')^2}{\kappa_1^2 + \kappa_2^2}$; or

iv) it is a Frenet curve of osculating order 4 with $\kappa_2 \neq \text{constant}$,

$\kappa_2 \neq \frac{c}{\sqrt{\kappa_3}}$,

$\frac{2 \kappa_2'}{\kappa_2} + \frac{\kappa_3'}{\kappa_3} = \frac{\kappa_2''}{\kappa_2} - \frac{\kappa_2 (\kappa_1^2 + \kappa_2^2 + \kappa_3^2)}{\kappa_2^2}$. 

and
\[(3.38) \quad \kappa_2'' \neq \kappa_2 \left( \kappa_1^2 + \kappa_2^2 + \kappa_3^2 \right) + \frac{3\kappa_2 (\kappa_2')^2}{\kappa_1^2 + \kappa_2^2}.\]

Here, \(c > 0\) is an arbitrary constant.

**Proof.** Let \(\gamma\) be of GAW(6)-type. Since \(\kappa_1 = \text{constant}\), by the use of equations (3.3), (3.4) and Theorem 3.1, we have
\[(3.39) \quad -3\kappa_1 \kappa_2 \kappa_2' = a_2 \kappa_1 + b_2 \left( -\kappa_1^2 - \kappa_1 \kappa_2^2 \right),\]
\[(3.40) \quad -\kappa_1^3 \kappa_2 - \kappa_1 \kappa_2^3 + \kappa_1 \kappa_2 \kappa_2' - \kappa_1 \kappa_2 \kappa_3^2 = b_2 \kappa_1 \kappa_2',\]
\[(3.41) \quad 2 \kappa_1 \kappa_2 \kappa_3 + \kappa_1 \kappa_2 \kappa_3' = b_2 \kappa_1 \kappa_2 \kappa_3.\]

If \(\kappa_1 = 0\), then \(\gamma\) is a straight line. Let \(d = 2\). Then \(\gamma\) is a circle and from (3.39), we obtain
\[a_2 - b_2 \kappa_1^2 = 0,\]
which is satisfied for some \(a_2, b_2\) non-zero differentiable functions. (3.40) and (3.41) are also satisfied. Now, let \(d = 3\). Then we have
\[(3.42) \quad -3\kappa_2 \kappa_2' = a_2 - b_2 \left( \kappa_1^2 + \kappa_2^2 \right),\]
\[(3.43) \quad \kappa_2'' = \kappa_2 (\kappa_1^2 + \kappa_2^2) = b_2 \kappa_2'.\]
Thus \(\kappa_2\) cannot be constant. So (3.42) and (3.43) give us
\[b_2 = \frac{\kappa_2''}{\kappa_2} - \frac{\kappa_2}{\kappa_2} (\kappa_1^2 + \kappa_2^2),\]
\[a_2 = -3 \kappa_2 \kappa_2' + \frac{\kappa_2''}{\kappa_2} (\kappa_1^2 + \kappa_2^2) - \frac{\kappa_2}{\kappa_2} (\kappa_1^2 + \kappa_2^2)^2,\]
both of which must be non-zero. Finally, let \(d = 4\). From (3.40), \(\kappa_2 \neq \text{constant}\). In this case, by the use of (3.39), (3.40) and (3.41), we obtain
\[(3.44) \quad b_2 = \frac{2 \kappa_2''}{\kappa_2} + \frac{\kappa_1'}{\kappa_2} = \frac{\kappa_2'}{\kappa_2} - \frac{\kappa_2}{\kappa_2} (\kappa_1^2 + \kappa_2^2 + \kappa_3^2),\]
\[(3.45) \quad a_2 = -3 \kappa_2 \kappa_2' + \frac{\kappa_2''}{\kappa_2} (\kappa_1^2 + \kappa_2^2) - \frac{\kappa_2}{\kappa_2} (\kappa_1^2 + \kappa_2^2) (\kappa_1^2 + \kappa_2^2 + \kappa_3^2).\]
Thus, from equation (3.44), we have
\[\kappa_2 \neq \frac{c}{\sqrt{\kappa_3}},\]
where \(c > 0\) is an arbitrary constant. We also have (3.38) from (3.45).

Converse proposition is done easily. \(\square\)

**Proposition 3.8.** Let \(\gamma \colon I \subseteq \mathbb{R} \to \mathbb{E}^n\) be a unit speed Frenet curve of osculating order \(d \leq 4\) with \(\kappa_1 = \text{constant}\). Then \(\gamma\) is of GAW(7)-type if and only if
\[i)\] it is a straight line; or
\[ii)\] it is a Frenet curve of osculating order 3 satisfying
\[\kappa_2 \neq \text{constant},\]
\[\kappa_2'' \neq \frac{3 \kappa_2 (\kappa_2')^2}{\kappa_1^2 + \kappa_2^2} + \kappa_2 (\kappa_1^2 + \kappa_2^2);\] or
iv) it is a Frenet curve of osculating order 4 satisfying
\[ \kappa_2 \neq \text{constant}, \]
\[ \kappa_2' \neq \frac{c}{\sqrt{\kappa_3}}, \]
\[ \frac{3\kappa_2\kappa_2'}{\kappa_1^2 + \kappa_2^2} = \frac{2\kappa_3'}{\kappa_2} + \frac{\kappa_3''}{\kappa_3}, \]
\[ \kappa_2'' \neq \frac{3\kappa_2 (\kappa_2')^2}{\kappa_1^2 + \kappa_2^2} + \kappa_2 (\kappa_1^2 + \kappa_2^2 + \kappa_3^2), \]
where \( c \) is an arbitrary constant.

Proof. Let \( \gamma \) be of GAW(7)-type. If we use equations (3.3), (3.4) and Theorem 3.1, we obtain
\[ (3.46) \quad -3\kappa_1\kappa_2\kappa_2' = b_3(-\kappa_1^3 - \kappa_1\kappa_2^2), \]
\[ (3.47) \quad -\kappa_1^3\kappa_2 - \kappa_1\kappa_2^3 + \kappa_2\kappa_2^3 - \kappa_1\kappa_2\kappa_3^2 = a_3\kappa_1\kappa_2 + b_2\kappa_1\kappa_2', \]
\[ (3.48) \quad 2\kappa_1\kappa_2\kappa_3 + \kappa_1\kappa_2\kappa_3' = b_2\kappa_1\kappa_2\kappa_3, \]
If \( d = 1 \), \( \gamma \) is a straight line. Let \( d = 2 \). Then, from (3.46), we find \( \kappa_1 = 0 \). This is a contradiction. Let \( d = 3 \). Then, using (3.46), \( \kappa_2 \) can not be constant. By the use of (3.46) and (3.47), we get
\[ (3.49) \quad b_3 = \frac{3\kappa_2\kappa_2'}{\kappa_1^2 + \kappa_2^2}, \]
\[ a_3 = \frac{\kappa_2''}{\kappa_2} - \frac{3\kappa_2 (\kappa_2')^2}{\kappa_2 (\kappa_1^2 + \kappa_2^2)} - (\kappa_1^2 + \kappa_2^2), \]
both of which are non-zero differentiable functions. Again, equation (3.49) requires \( \kappa_2 \) is not a constant. We also have
\[ \kappa_2'' \neq \frac{3\kappa_2 (\kappa_2')^2}{\kappa_1^2 + \kappa_2^2} + \kappa_2 (\kappa_1^2 + \kappa_2^2). \]
Now, let \( d = 4 \). Then, using equations (3.46), (3.47) and (3.48), we obtain
\[ (3.50) \quad b_3 = \frac{3\kappa_2\kappa_2'}{\kappa_1^2 + \kappa_2^2} = \frac{2\kappa_3'}{\kappa_2} + \frac{\kappa_3''}{\kappa_3}, \]
\[ a_3 = \frac{\kappa_2''}{\kappa_2} - \frac{3\kappa_2 (\kappa_2')^2}{\kappa_2 (\kappa_1^2 + \kappa_2^2)} - (\kappa_1^2 + \kappa_2^2 + \kappa_3^2), \]
which give us
\[ \kappa_2 \neq \text{constant}, \]
\[ \kappa_2' \neq \frac{c}{\sqrt{\kappa_3}}, \]
\[ \kappa_2'' \neq \frac{3\kappa_2 (\kappa_2')^2}{\kappa_1^2 + \kappa_2^2} + \kappa_2 (\kappa_1^2 + \kappa_2^2 + \kappa_3^2). \]
Here, \( c \) is an arbitrary constant.

Converse proposition is trivial. \( \square \)
References


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