ON A SPECIAL CONFIGURATION OF LINES AND POINTS IN $\mathbb{P}^N$

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Abstract. This note concerns some arrangements of lines in $\mathbb{P}^N(\mathbb{C})$ and the condition under which there exists a hyperplane intersecting transversely every line of the given arrangement at a unique point.

1. Introduction.

In this note we want to address the following combinatorial problem. Let us fix a set $\mathcal{L}$ of $r$ disjoint lines $\{L_1, L_2, \ldots, L_r\}$ in $\mathbb{P}^N(\mathbb{C})$. Let us pick $r$ distinct points $\{P_1, \ldots, P_r\}$ such that $P_i \in L_i$ for $i = 1, \ldots, r$. Under which conditions can one find a hyperplane through $P_1, \ldots, P_r$ that intersects each line $L_i$ exactly at $P_i$? It easy to see that there are situations in which no such hyperplane exists. For instance, let $\langle \ldots \rangle$ denote the linear span of a given subset, and assume that $\dim(\langle L_1, L_2, L_3, L_4 \rangle) = 3 < N$. Then, for any generic 4-tuple of points $P_1, P_2, P_3, P_4$, chosen respectively on $L_1, L_2, L_3, L_4$, every hyperplane in $\mathbb{P}^N$ that contains all of these points must also contain all of the lines $L_1, L_2, L_3, L_4$.

The above example suggests that the dimension of the linear spans of subsets of $\mathcal{L}$ play a significant role, and that with no additional assumptions on such dimensions one cannot hope to find a general solution. However, if we assume that for any subset $\mathcal{L}' \subseteq \mathcal{L}$ the dimension of the corresponding linear span depends only upon the cardinality of $\mathcal{L}'$, a suitable general result can be achieved. As we shall see, the hypothesis above is satisfied, for instance, when $\mathcal{L}$ is any subset of $r \leq N$ fibres of a rational scroll embedded in $\mathbb{P}^N(\mathbb{C})$.

In Section 2 the main theorem is presented. Section 3 contains a few corollaries showing that, in the situation under consideration, the set of hyperplanes in $\mathbb{P}^{N*}$ (the dual space) satisfying the main hypothesis above with respect to $\mathcal{L}$ is large enough. Section 4 studies the special case in which all lines $L_i$ are contained in a rational ruled surface. Finally Section 5 is devoted to an application of the main theorem which was indeed the original motivation for us to address this problem.

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2. The main theorem

Let us consider a set \( \mathcal{L} \) as mentioned in §1. Let us fix a line \( L := L_1 \in \mathcal{L} \). Let us pick a second line \( L_2 \) such that \( \dim((L_1, L_2)) = 3 \). Then, let us pick a third line \( L_3 \), if it exists, such that \( \dim((L_1, L_2, L_3)) = 5 \), and so on. We can proceed in this way, say, only for \( h \geq 1 \) steps to get \( L_1, L_2, ..., L_{h+1} \) with \( \dim((L_1, L_2, ..., L_{h+1})) = 2h + 1 \leq N \). Now we pick another line \( L_{h+2} \), if it exists, such that \( L_{h+2} \) intersects \( (L_1, L_2, ..., L_{h+1}) \) at one point only; then we pick another line \( L_{h+3} \), if it exists, intersecting \( (L_1, L_2, ..., L_{h+1}, L_{h+2}) \) at one point only, and so on. If possible, we can proceed in this way, say, only for another \( q \geq 1 \) steps to get \( L_1, L_2, ..., L_{h+1}, L_{h+2}, ..., L_{h+q+1} \) with \( \dim((L_1, L_2, ..., L_{h+1}, L_{h+2}, ..., L_{h+q+1})) = 2h + q + 1 = N \). Then, independently of the number of the remaining lines, if any, \( \dim((L_1, L_2, ..., L_{h+1}, L_{h+2}, ..., L_{h+q+1}, ..., L_p)) = N \) for any \( h + q + 2 \leq p \leq r \).

Notice that the function \( d : [1, r] \subseteq N \to N \) such that \( d(n) = \dim((L_1, L_2, ..., L_n)) \) depends upon the order in which our lines were chosen. Here we want to consider only sets \( \mathcal{L} \) of \( r \) lines in \( \mathbb{P}^N \), \( r \leq N \), such that \( d \) does not depend upon the order. In this case we can prove the following theorem, where \( k := h + q \).

**Theorem 2.1.** Let \( (h, k) \) be a given pair of integers with \( 1 \leq h \leq k \), \( h + k + 1 \equiv N \). Let \( \mathcal{L} = \{L_1, \ldots, L_r\} \), with \( 2 \leq r \leq N \), be any set of \( r \) distinct and disjoint lines in \( \mathbb{P}^N \), such that, for any subset \( \{L_1, \ldots, L_p\} \subseteq \mathcal{L} \), \( (p \leq r) \), one has:

\[
\begin{align*}
1) & \dim((L_1, \ldots, L_p)) = 2p - 1 \text{ when } 1 \leq p \leq h + 1; \\
2) & \dim((L_1, \ldots, L_p)) = p + h \text{ when } h + 2 \leq p \leq k + 1; \\
3) & \dim((L_1, \ldots, L_p)) = N \text{ when } k + 2 \leq p \leq N.
\end{align*}
\]

Let \( W_r := \{(P_1, \ldots, P_r) \in L_1 \times \cdots \times L_r | (\mathbb{P}^1)^r \cap \dim((P_1, \ldots, P_r)) \leq r - 2\}. \)

Then \( \dim(W_r) \leq r - 2 \), i.e. \( W_r \) is a closed subscheme of codimension at least \( 2 \) in \( (\mathbb{P}^1)^r \subseteq \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \) (\( r \) times). Moreover, if \( 2 \leq r \leq h + 1 \) then \( W_r \) is empty, if \( h + 2 \leq r \leq k + 1 \) then \( \dim(W_r) \leq r - h - 2 \).

Before proving Theorem 2.1 we would like to show that there are concrete situations in which the assumptions of Theorem 2.1 are indeed satisfied.

**Lemma 2.1.** Let \( \mathcal{E} = O_{\mathbb{P}^1}(h) \oplus O_{\mathbb{P}^1}(k) \) with \( 1 \leq h \leq k \), \( N = h + k + 1 \), and let \( S = \mathbb{P}(\mathcal{E}) \), be a smooth, rational, surface embedded as a linear scroll in \( \mathbb{P}^N(\mathbb{C}) \) by its tautological line bundle. Let \( L_1, L_2, ..., L_r \) be any set of \( r \) lines in \( \mathbb{P}^N \) which are fibres of the scroll \( S \), with \( 2 \leq r \leq N \). Then all the assumptions of Theorem 2.1 hold for \( L_1, L_2, ..., L_r \).

**Proof.** Let \( T \) be the very ample tautological divisor of \( S \). Let \( f_{H_1}, \ldots, f_{H_r} \) be the \( r \) fibres of \( S \), over the points \( H_1, \ldots, H_r \), of the base curve \( C \cong \mathbb{P}^1 \), corresponding to \( L_1, L_2, ..., L_r \). If we consider the linear space of \( \mathbb{P}^N \) spanned by any subset of \( \rho \) lines in \( \{L_1, L_2, ..., L_r\} \), corresponding to \( \rho \) points in \( \{H_1, ..., H_r\} \), say \( H_1, ..., H_\rho \), we have that its dimension is

\[
N - h^0(S, T - f_{H_1} - \cdots - f_{H_\rho}) = N - h^0(C, \mathcal{E} \otimes O_C(-H_1 - \cdots - H_\rho)) = N - h^0(\mathbb{P}^1, O_{\mathbb{P}^1}(h - \rho))
\]

Now, if \( 1 \leq \rho \leq h \) the dimension is \( N - (h - \rho + 1 + k - \rho + 1) = 2\rho - 1 \). If \( h < \rho \leq k \) the dimension is \( N - (k - \rho + 1) = \rho + h \). If \( k < \rho \) the dimension is \( N \).
In other words:
\[
\dim(\langle L_1, \ldots, L_\rho \rangle) = \begin{cases} 
2\rho - 1 & \text{if } 1 \leq \rho \leq h + 1 \\
\rho + h & \text{if } h + 2 \leq \rho \leq k + 1 \\
N & \text{if } k + 2 \leq \rho \leq N.
\end{cases}
\]

Hence assumptions 1), 2), 3) of Theorem 2.1 hold for \(L_1, L_2, \ldots, L_r\).

The following remark will be very useful for the proof of Theorem 2.1.

**Remark 2.1.** Let \(\mathcal{L}\) be a set of lines in \(\mathbb{P}^N\) satisfying the assumptions of Theorem 2.1. Let \(\mathcal{L}' = \{L_1, \ldots, L_r\} \subseteq \mathcal{L}\) be any subset of \(\mathcal{L}\), with \(2 \leq r' \leq r\), having a corresponding subscheme \(W_{r'}\), defined similarly as in Theorem 2.1. Note that \(\mathcal{L}'\) satisfies the same assumptions as \(\mathcal{L}\), so that to prove Theorem 2.1 one can proceed by induction on \(r'\), assuming that \(\dim(W_{r'}) \leq r' - 2\) for any \(\mathcal{L}' \subseteq \mathcal{L}\) with \(r' \leq r\), we will show that \(\dim(W_r) \leq r - 2\).

As suggested by Remark 2.1, the proof of Theorem 2.1 will proceed by induction on \(r\), and will make use of a few preliminary Lemmata. The following Lemma collects two simple observations that will facilitate the induction process.

**Lemma 2.2.** In the assumptions of Theorem 2.1, let \(r \geq 3\) and let \(m\) be any fixed positive integer. Assume that \(\dim(W_{r'}) \leq r' - m\) for any subset of \(r' < r\) lines in \(\mathcal{L}\). Then, in order to prove that \(\dim(W_r) \leq r - m\), one can assume that for any generic configuration \((P_1, \ldots, P_r) \in L_1 \times L_2 \times \cdots \times L_r \simeq (\mathbb{P}^1)^{\times r}\) in \(W_r\), the following facts are true:

1) \(\dim(\langle P_1, \ldots, P_r \rangle) = r - 2\)

2) \(\dim(\langle \hat{P}_1, \ldots, \hat{P}_i, \ldots, P_r \rangle) = r - 2\) for any \(i\), where \(\hat{P}_i\) is deleted.

**Proof.** To prove that we can assume 1), let us consider \(W'_r := \{(P_1, \ldots, P_r) \in L_1 \times L_2 \times \cdots \times L_r \simeq (\mathbb{P}^1)^{\times r}\} \dim(\langle P_1, \ldots, P_r \rangle) \leq r - 3\) \(\subseteq W_r\) (if \(r = 3\) \(W'_r = \emptyset\)).

If we project any \(r\)-uple of \(W'_r\) onto any product of \(r - 1\) lines chosen in \(\mathcal{L}\) we get a \((r - 1)\)-tuple of the set \(W_{r-1}\) corresponding to those \(r - 1\) lines. By assumption \(\dim(W_{r-1}) \leq r - 1 - m\), hence \(\dim(W'_r) \leq r - 1 - m + 1 = r - m\). Therefore if \(W_r = W'_r\) then \(\dim(W_r) \leq r - m\), so that we can always assume that \(W_r \supseteq W'_r\), i.e. fact 1).

To prove that we can assume 2), choose any \(i \in \{1, \ldots, r\}\) and let us consider the closed subscheme \(W_{r-1}\) corresponding to the subset \(\{L_1, \ldots, \hat{L}_i, \ldots, L_r\} \subseteq \mathcal{L}\), where \(\hat{L}_i\) is removed. Obviously \(W_{r-1} \times L_i \subseteq W_r\). By assumption \(\dim(W_{r-1}) \leq r - 1 - m\), hence \(\dim(W_{r-1} \times L_i) \leq r - 1 - m + 1 = r - m\). Therefore if \(W_r = W_{r-1} \times L_i\) then \(\dim(W_r) \leq r - m\), so that we can always assume that \(W_r \supseteq W_{r-1} \times L_i\). As this is true for any \(i \in \{1, \ldots, r\}\) we can assume fact 2).

**Lemma 2.3.** Let \(L_1, \ldots, L_{h+1}\) be disjoint lines in \(\mathbb{P}^{2h+1}\), with \(h \geq 1\), such that their linear span has maximal dimension, i.e. \((L_1, \ldots, L_{h+1}) = \mathbb{P}^{2h+1}\). For any \(Q \in \mathbb{P}^{2h+1}\) let \(t_Q \leq h + 1\) be the minimum number of lines among the \(L_i's\) necessary to have \(Q\) contained in their linear span, which has dimension \(2t_Q - 1\). Let \(W_{h+1}(Q) := \{(P_1, \ldots, P_{h+1}) \in L_1 \times L_2 \times \cdots \times L_{h+1} \simeq (\mathbb{P}^1)^{\times (h+1)}\} \dim(\langle Q, P_1, \ldots, P_{h+1} \rangle) \leq h\). Then \(0 \leq \dim(W_{h+1}(Q)) \leq h + 1 - t_Q\) and if \(\dim(W_{h+1}(Q)) = 0\) then \(W_{h+1}(Q)\) is a single point.
Proof. The proof will be conducted in detail for $h = 2$. The general case is handled exactly in the same fashion. As the given lines have a linear span of maximal dimension, it is possible to choose a coordinate system in the ambient space $\mathbb{P}^{2h+1=5}$ such that its $2h + 2 = 6$ fundamental points belong, pairwise, to the $h + 1 = 3$ given lines. In this situation, let us consider the $(h + 2 = 4, 2h + 2 = 6)$ matrix $M$ whose first $3 = h + 1$ rows are given by the coordinates of points on the lines $L_1, L_2, L_3$, and where the last row consists of the coordinates of $Q$:

$$
M = \begin{bmatrix}
\alpha_1 & \beta_1 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_2 & \beta_2 & 0 & 0 \\
x_0 & x_1 & x_2 & x_3 & x_4 & x_5 
\end{bmatrix},
$$

For any $Q$, $W_3(Q)$ is given by all possible choices of pairs $(\alpha_i : \beta_i) \neq (0,0)$ for which $\text{rk}(M) \leq 3$. It is easy to see that, for any $Q$, there exists at least one such choice of pairs $(\alpha_i : \beta_i)$, namely $(\alpha_i : \beta_i) = (x_{2i-2}, x_{2i-1})$ for all pairs $(x_{2i-2}, x_{2i-1}) \neq (0,0)$, hence $\dim(W_3(Q)) \geq 0$.

To get the other side of the stated inequality notice that, as $(\alpha_i : \beta_i) \neq (0 : 0)$, $M$ can always be transformed into the following matrix

$$
M_1 = \begin{bmatrix}
1 & 0 & 0 & \lambda_1 & 0 & 0 \\
0 & 1 & 0 & 0 & \lambda_2 & 0 \\
y_0 & y_1 & y_2 & y_3 & y_4 & y_5 
\end{bmatrix},
$$

where $\text{rk}(M_1) = \text{rk}(M)$, $\lambda_i = \alpha_i/\beta_i$ or $\lambda_i = \beta_i/\alpha_i$ respectively when $\beta_i \neq 0$ or $\alpha_i \neq 0$, and $(y_0, \ldots, y_5)$ is a permutation of $(x_0, \ldots, x_5)$. $M_1$ can then be further transformed, keeping its rank unaltered:

$$
M_2 = \begin{bmatrix}
1 & 0 & 0 & \lambda_1 & 0 & 0 \\
0 & 1 & 0 & 0 & \lambda_2 & 0 \\
0 & 0 & 1 & 0 & 0 & \lambda_3 \\
y_0 - \lambda_1 y_0 & y_3 - \lambda_2 y_2 & y_5 - \lambda_3 y_4 
\end{bmatrix}
$$

It is $\text{rk}(M_2) \leq 3$ if and only if:

$$
\begin{cases}
\lambda_1 y_0 = y_1 \\
\lambda_2 y_2 = y_3 \\
\lambda_3 y_4 = y_5.
\end{cases}
$$

As $\dim(W_3(Q)) \geq 0$, the above system must have at least one solution. If no equation is identically satisfied, then there exists only one solution $(\lambda_1, \lambda_2, \lambda_3)$, corresponding to a triplet of points, one for each line; in this case $\dim(W_3(Q)) = 0$ and the Lemma is proved. If there is only one identically satisfied equation, say $y_0 = y_1 = 0$, then $\dim(W_3(Q)) = 1$ (you can choose an arbitrary point on the first line, but then the other two are determined) and in this case $Q \in \langle L_2, L_3 \rangle$, hence $t_Q = 2$ and the Lemma is proved. If exactly two equations are identically satisfied, say $y_0 = y_1 = y_2 = y_3 = 0$, then $\dim(W_3(Q)) = 2$ (you can choose arbitrary points on the first two lines, while the last point is uniquely determined), and in this case $Q \in \langle L_3 \rangle$ hence $t_Q = 1$ and the Lemma is proved. As $(y_0, \ldots, y_5)$ is a permutation of projective coordinates of $Q$, not all equations can be identically satisfied, so that the Lemma is proved for $h = 2$. \qed
Lemma 2.4. Under the assumptions of Theorem 2.1, further assume that \( r \geq h + 2 \). Let \( \mathcal{L} = \{L_1', L_2', \ldots, L_{h+1}'\} \) be any subset of \( h + 1 \) lines chosen from the given set \( \mathcal{L} = \{L_1, \ldots, L_r\} \). Let \( L \in \mathcal{L} \setminus \mathcal{L}' \). Then \( L \) intersects the \((2h + 1)\)-dimensional linear space \( \langle L_1', L_2', \ldots, L_{h+1}' \rangle \) only at one point \( Q \) and such \( Q \) does not belong to any linear space spanned by any proper subset of \( \mathcal{L}' \). Moreover, there exists a unique choice of \( P_i \in L_i' \), such that \( \dim(\langle Q, P_1, \ldots, P_{h+1} \rangle) \leq h \).

Proof. Assumption 1) of Theorem 2.1 gives that \( \mathcal{L}' \) spans a \((2h + 1)\)-dimensional linear subspace. Any other line \( L \in \mathcal{L} \setminus \mathcal{L}' \), cuts this subspace only at one point \( Q \), by assumption 2). Moreover, \( Q \) can not belong to any linear space spanned by a proper subset of \( \mathcal{L}' \), otherwise the union of this proper subset and \( L \) would contradict assumption 1) of Theorem 2.1. Therefore Lemma 2.3, gives a unique choice of \( P_i \in L_i' \) such that \( \dim(\langle Q, P_1, \ldots, P_{h+1} \rangle) \leq h \). \( \square \)

The above Lemmata will now be combined to provide a proof for Theorem 2.1.

Proof. (of Theorem 2.1).

It is convenient to divide the proof into 4 cases, according to the relative sizes of \( r, h \) and \( k \).

**Case 1:** \( 2 \leq r \leq h + 1 \). In this case \( W_r \) is actually empty. To see this, choose a coordinate system in \( \mathbb{P}^N \) such that \( 2r \) points among its \( N + 1 \) fundamental points belong, pairwise, to the \( r \) given lines. This is possible by assumption 1). As in the proof of Lemma 2.3, consider the following \((r, N+1)\) matrix whose rows are given by the coordinates of points on each of the given \( r \) lines:

\[
\begin{pmatrix}
\alpha_1 & \beta_1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & \alpha_2 & \beta_2 & \ldots & 0 & 0 & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & \alpha_r & \beta_r & 0 & \ldots & 0
\end{pmatrix}.
\]

As \((\alpha_i : \beta_i) \neq (0 : 0)\) for all \( i \), it is clear that there always exists a non singular, rank \( r \), submatrix and thus \( W_r = \emptyset \).

**Case 2:** \( 2 \leq r \leq k + 1 \). In this case, induction on \( r \) will show that

\[
\dim(W_r) \leq r - h - 2.
\]

This slightly stronger inequality implies the statement and it will be useful in proving the remaining cases. If \( k = h \), or \( 2 \leq r \leq h + 1 \), there is nothing to prove after Case 1, so we can assume \( h < k \) and \( r \geq h + 2 \) (note that this implies \( r \geq 3 \)). Our inductive hypothesis is that the desired inequality (2.1) holds for each subset of \( r' \) lines contained in \( \mathcal{L} \), with \( 2 \leq r' < r \) (recall remark 2.1). Moreover, for a generic \((P_1, \ldots, P_r) \in W_r\), it is enough to consider the cases that \( \dim(\langle P_1, \ldots, P_r \rangle) = r - 2 \) by part 1) of Lemma 2.2.

Fix any order for \( \mathcal{L} \) and, recalling Lemma 2.4, let \( Q_1, Q_2, \ldots, Q_{r-h-1} \) be the points of intersection of each of the last \( r - h - 1 \) lines with the linear subspace spanned by the first \( h + 1 \) lines. Let us choose a coordinate system in \( \mathbb{P}^N \) such that its first \( 2(h+1) \) fundamental points belong, pairwise, to the first \( h+1 \) lines, and such that each one of the remaining fundamental points belongs to one of the remaining lines. Notice that these remaining fundamental points are certainly distinct from \( Q_1, Q_2, \ldots, Q_{r-h-1} \). The rows of the following \((r, N+1)\) matrix \( M \) are given by the coordinates of the points of the \( r \) given lines:
This fact implies that in any matrix
\[ \dim(Q) \leq r \]
without loss of generality, let us assume that the maximal dimension is \( W \) to the linear subspace spanned by any proper subset of \( \text{rk}(W) = M \leq h \). Moreover, as for the generic\footnote{Remark 2.1. Having established Cases 1 and 2 we can assume that \( \text{rk}(W) = M \leq h + 1 \)}\footnote{Case 3: \( 2 \leq r \leq N \) and \( 1 \leq r - (h + 1) \leq h + 1 \). In this case, inequality \( \dim(W) \leq r - 2 \) will be established by induction on \( r \), keeping always in mind Remark 2.1. Having established Cases 1 and 2 we can assume that \( k + 2 \leq r \leq N \) and, by Lemma 2.2 part 1), we can also assume that \( \dim(P_{1}, \ldots, P_{r}) = r - 2 \) for the generic \( r \)-tuple of \( W \). Fix an order for \( L \) and let us divide any \( r \)-tuple in \( W \) into two non empty subsets: \( (P_{1}, \ldots, P_{r}) = (P_{1} \ldots, P_{h+1})(P_{h+2}, \ldots, P_{r}) \). As \( (P_{1}, \ldots, P_{r}) \in W \) we have: \( \dim(P_{1}, \ldots, P_{h+1}) \leq r - 2 \) and, by Case 1, \( \dim(P_{1}, \ldots, P_{h+1}) = h \) and \( \dim(P_{h+2}, \ldots, P_{r}) = r - (h + 1) - 1 \). Hence \( \dim(P_{1}, \ldots, P_{h+1}) \cap (P_{h+2}, \ldots, P_{r}) \geq h + r - (h + 1) - 1 = r + 2 \). Therefore there always exists at least a point \( Q \in (P_{1}, \ldots, P_{h+1}) \cap (P_{h+2}, \ldots, P_{r}) \). Moreover, as for the generic \( r \)-tuple of \( W \) it is true that \( \dim(P_{1}, \ldots, P_{r}) = \dim(P_{1}, \ldots, P_{h+1}) \cup (P_{h+2}, \ldots, P_{r}) = r - 2 \) which, as for the generic \( r \)-tuple of \( W \) it is true that \( \dim(P_{1}, \ldots, P_{r}) = \dim(P_{1}, \ldots, P_{h+1}) \cup (P_{h+2}, \ldots, P_{r}) = r - 2 \) we can also say that for the generic \( r \)-tuple of \( W \) there exists a unique point \( Q \in (P_{1}, \ldots, P_{h+1}) \cap (P_{h+2}, \ldots, P_{r}) \). Now, let us consider \( (L_{1}, \ldots, L_{h+1}) \) and \( (L_{h+2}, L_{h+3}, \ldots, L_{r}) \) and let us consider \( (L_{1}, \ldots, L_{h+1}) \) and \( (L_{h+2}, L_{h+3}, \ldots, L_{r}) \). As \( 1 \leq r - (h + 1) \leq h + 1 \) by assumption 1) we can say that \( \dim(L_{1}, \ldots, L_{h+1}) = 2(h + 1) - 1 \) and \( \dim(L_{h+2}, L_{h+3}, \ldots, L_{r}) = 2(r - h - 1) - 1 \). As \( k + 2 \leq r \leq N \) we can say that \( \dim(L_{1}, \ldots, L_{h+1}) \cup (L_{h+2}, L_{h+3}, \ldots, L_{r}) = \dim(L_{1}, \ldots, L_{r}) = N \). Hence, if we define \( A := (L_{1}, \ldots, L_{h+1}) \cap (L_{h+2}, L_{h+3}, \ldots, L_{r}) \), we have that \( \dim(A) = 2h + 1 + 2r - 2h - 3 - N = 2r - 2 - N \leq r - 2 \). Moreover, as we saw that}
for the generic $r$-tuple of $W_r$, there exists a (unique) point $Q \in \langle P_1, \ldots, P_{h+1} \rangle \cap \langle P_{h+2}, \ldots, P_r \rangle \subseteq A$, we can also say that $A$ is not empty (unless $W_r$ is empty, in which case there is nothing to prove). The linear space $A$ contains all intersections of lines $L_{h+1}, L_{h+2}, \ldots, L_r$ with $\langle L_1, \ldots, L_{h+1} \rangle$, and these intersection points surely exist by assumption 2). Therefore $A$ can not be contained in a linear subspace of $\langle L_1, \ldots, L_{h+1} \rangle$ spanned by a proper subset of these lines because no one of those points belong to such a space, thanks to Lemma 2.4. Lemma 2.3 then implies that, for a generic $Q \in A$, there exists a unique $(h+1)$-tuple of points $P_1, \ldots, P_{h+1}$, such that $\dim(\langle Q, P_1, \ldots, P_{h+1} \rangle) \leq h$.

Let us introduce in $L_1 \times L_2 \times \cdots \times L_r \times A \simeq (\mathbb{P}^1)^r \times \mathbb{P}^{2r-2-N}$ the following (non empty) incidence variety:

$$J := \{(P_1, \ldots, P_r, Q) \in (\mathbb{P}^1)^r \times A | Q \in \langle P_1, \ldots, P_{h+1} \rangle \cap \langle P_{h+2}, \ldots, P_r \rangle \}
= \{(P_1, \ldots, P_r, Q) \in (\mathbb{P}^1)^r \times A | \dim(\langle Q, P_1, \ldots, P_{h+1} \rangle) \leq h
\text{ and } \dim(Q, P_{h+2}, \ldots, P_r) \leq r - (h + 1) - 1 \}.$$

Let $p: J \to (\mathbb{P}^1)^r$ and $f: J \to A$ be the natural projections. It is $p(J) \subseteq W_r$ because if $(P_1, \ldots, P_r, Q) \in J$ then the points $(P_1, \ldots, P_r)$ can not be linearly independent in $\mathbb{P}^N$. On the other hand we have seen that for any $r$-tuple of $W_r$ there exist at least a point $Q \in \langle P_1, \ldots, P_{h+1} \rangle \cap \langle P_{h+2}, \ldots, P_r \rangle \subseteq A$ and that for the generic $r$-tuple of $W_r$ there exist a unique point $Q$. Hence $\dim(J) = \dim(W_r)$ and $\dim(J) = \dim(W_r)$. Then $\dim(W_r) = \dim(J) = \dim(\text{Im}(f)) + \dim(\text{generic fibre of } f)$.

Let us consider any point $Q \in A$. As $Q \in \langle L_1, \ldots, L_{h+1} \rangle$, Lemma 2.3 implies that there exists at least an $(h+1)$-tuple of points $(P_1, \ldots, P_{h+1})$ such that $\dim(\langle Q, P_1, \ldots, P_{h+1} \rangle) \leq h$. As $Q \in \langle L_{h+2}, L_{h+3}, \ldots, L_r \rangle$, Lemma 2.3 implies that there exists at least an $(r-h-1)$-tuple of points $(P_{h+2}, \ldots, P_r)$ such that $\dim(\langle Q, P_{h+2}, \ldots, P_r \rangle) \leq r - h - 2$. Therefore $\text{Im}(f) = A$.

In order to estimate the dimension of a generic fiber of $f$, let $Q$ be now a generic point of $A$. Lemma 2.3 implies that $A$ can not be contained in a linear subspace of $\langle L_1, \ldots, L_{h+1} \rangle$ spanned by a proper subset of these lines and that there exists a unique $(h+1)$-tuple of points $(P_1, \ldots, P_{h+1})$ such that $\dim(\langle Q, P_1, \ldots, P_{h+1} \rangle) \leq h$. Hence to get a bound for $\dim(f^{-1}(Q))$ it suffices to consider the $(r-h-1)$-tuples of points $P_{h+2}, \ldots, P_r$ such that $\dim(\langle Q, P_{h+2}, \ldots, P_r \rangle) \leq r - h - 2$. With the notation introduced in the proof of Lemma 2.3, it is true that $\dim(f^{-1}(Q)) = \dim(W_{r-h-1}(Q)) = \dim(\langle P_{h+2}, \ldots, P_r \rangle) \in \langle L_{h+1} \times L_{h+2} \times \cdots \times L_r \rangle \dim(\langle Q, P_{h+2}, \ldots, P_r \rangle) \leq r - h - 2$).

If $A$ is not contained in a linear subspace of $\langle L_{h+1}, \ldots, L_r \rangle$ spanned by a proper subset of these lines, Lemma 2.3 gives that for the generic point $Q \in A$ there exists only one $(r-h-1)$-tuple of points $(P_{h+2}, \ldots, P_r)$ such that $\dim(\langle Q, P_{h+2}, \ldots, P_r \rangle) \leq r - h - 2$. In this case $\dim(f^{-1}(Q)) = 0$ and therefore $\dim(W_r) = \dim(\text{Im}(f)) + \dim(\text{generic fibre of } f) = \dim(A) = 2r - 2 - N \leq r - 2$ and we are done.

If $A$ is contained in at least one linear subspace spanned by a proper subset of $\langle L_{h+1}, \ldots, L_r \rangle$, let $2t - 1$ be the dimension of the space, spanned by $t$ lines, with the minimal dimension among them. Note that $1 \leq t < r - (h + 1) \leq h + 1$. For all $Q \in A$ Lemma 2.3 gives $\dim(W_{r-h-1}(Q)) \leq r - h - 1 - t$. Then we have
\[ \dim(W_r) = \dim(J) = \dim(\text{Im}(f)) + \dim(\text{generic fibre of } f) \leq 2t - 1 + r - h - 1 - t = t + r - h - 2 < h + 1 + r - h - 2 = r - 1, \text{ i.e. } \dim(W_r) \leq r - 2.\]

**Case 4:** \(2 \leq r \leq N\) and \(h + 2 \leq r - (h + 1) < k + 1\). Because of Cases 1, 2 and 3 we can assume \(k + 2 \leq r \leq N\) and, by Lemma 2.2 part 1), we can also assume that \(\dim((P_1, \ldots, P_r)) = r - 2\) for the generic \(r\)-tuple of \(W_r\). From Case 2 we have \(\dim(W_{r-h-1}) \leq r - h - 1 - h - 2 = r - 2h - 3\).

As before, fix an order for \(L\) and let us divide every \(r\)-tuple in \(W_r\) into two non-empty subsets \((P_1, \ldots, P_r) = (P_1, \ldots, P_{h+1})(P_{h+2}, \ldots, P_r)\). As \((P_1, \ldots, P_r) \in W_r\) we have that \(\dim((P_1, \ldots, P_{h+1}) \cup (P_{h+2}, \ldots, P_r)) \leq r - 2\) and, from Case 1, \(\dim((P_1, \ldots, P_{h+1})) = h\). Moreover, Lemma 2.2 part 2) gives \(\dim((P_{h+2}, \ldots, P_r)) = r - h - 2\) for the generic \(r\)-tuple of \(W_r\).

Thus \(\dim((P_1, \ldots, P_{h+1}) \cap (P_{h+2}, \ldots, P_r)) \geq h + r - h - 2 - r + 2 = 0\) and therefore there always exists at least one point \(Q \in (P_1, \ldots, P_{h+1}) \cap (P_{h+2}, \ldots, P_r)\). Moreover, as \(\dim((P_1, \ldots, P_{h+1})) = \dim((P_{h+2}, \ldots, P_r)) = r - 2\), for a generic \(r\)-tuple of \(W_r\), it follows that there exists a unique point \(Q \in (P_1, \ldots, P_{h+1}) \cap (P_{h+2}, \ldots, P_r)\).

As in the previous case let us consider \((L_1, \ldots, L_{h+1})\) and \((L_{h+2}, \ldots, L_r)\). As \(h + 2 \leq r - (h + 1) < k + 1\) by assumptions 1) and 2) we have \(\dim((L_1, \ldots, L_{h+1})) = 2(h + 1) - 1\) and \(\dim((L_{h+2}, L_{h+3}, \ldots, L_r)) = r - h - 1 + h = r - 1\). As \(h + 2 \leq r \leq N\) we have \(\dim((L_1, \ldots, L_{h+1}) \cup (L_{h+2}, L_{h+3}, \ldots, L_r)) = \dim((L_1, \ldots, L_r)) = N\). As in the previous case, let \(A = (L_1, \ldots, L_{h+1}) \cap (L_{h+2}, L_{h+3}, \ldots, L_r)\). It is \(\dim(A) = 2h + 1 + r - 1 - N = 2h + r - N\). Moreover, as for the generic \(r\)-tuple of \(W_r\) there exists a (unique) point \(Q \in (P_1, \ldots, P_{h+1}) \cap (P_{h+2}, \ldots, P_r)\) the following (non-empty) incidence variety:

\[
J : = \{(P_1, \ldots, P_r, Q) \in (\mathbb{P}^1)^{\times r} \times A | Q \in (P_1, \ldots, P_{h+1}) \cap (P_{h+2}, \ldots, P_r)\}
\]

is \(\dim((Q, P_1, \ldots, P_{h+1})) \leq h\).

As in the previous case, let us introduce in \(L_1 \times L_2 \times \cdots \times L_r \times A \simeq (\mathbb{P}^1)^{\times r} \times \mathbb{P}^{2h+r-N}\) the following (non-empty) incidence variety:

\[
J : = \{(P_1, \ldots, P_r, Q) \in (\mathbb{P}^1)^{\times r} \times A | \dim((Q, P_1, \ldots, P_{h+1})) \leq h, \dim((Q, P_{h+2}, \ldots, P_r)) \leq r - (h + 1) - 1\}.
\]

Let \(p : J \rightarrow (\mathbb{P}^1)^{\times r}\) and \(f : J \rightarrow A\) be the natural projections. Note that \(p(J) \subseteq W_r\) because if \((P_1, \ldots, P_r, Q) \in J\) then \((P_1, \ldots, P_r)\) are not linearly independent in \(\mathbb{P}^N\). On the other hand we have seen that for every \(r\)-tuple of \(W_r\) there exists at least a point \(Q \in (P_1, \ldots, P_{h+1}) \cap (P_{h+2}, \ldots, P_r) \subseteq A\) and that for the generic \(r\)-tuple of \(W_r\) there exists a unique such \(Q\). Hence \(\text{Im}(p) = W_r\) and \(\dim(J) = \dim(W_r)\). Then \(\dim(W_r) = \dim(J) = \dim(\text{Im}(f)) + \dim(\text{generic fibre of } f) \leq \dim(A) + \dim[f^{-1}(\mathcal{O})] = 2h + r - N + \dim[f^{-1}(\mathcal{O})]\) where \(\mathcal{O}\) is now any fixed point of \(\text{Im}(f)\). Pick \(\mathcal{O} := (L_1, \ldots, L_{h+1}) \cap L_{h+2}\). Obviously \(\mathcal{O} \in A\). Moreover, as \(\mathcal{O}\) is the intersection point of \(L_{h+2}\) with \((L_1, \ldots, L_{h+1})\), we know that it does not belong to any linear subspace of \((L_1, \ldots, L_{h+1})\) spanned by a proper subset of these lines. Hence there exists a unique \((h + 1)\)-tuple of points \(P_1, \ldots, P_{h+1}\), such that \(\dim((\mathcal{O}, P_1, \ldots, P_{h+1})) \leq h\). Choosing \(P_{h+2} = \mathcal{O}\) one sees that there
exists also a \((r - h - 1)\)-tuple of points \((P_{h+2}, \ldots, P_r) \in L_{h+2} \times L_{h+3} \times \cdots \times L_r\) such that \(\dim((Q, P_{h+2}, \ldots, P_r)) \leq r - (h + 1) - 1\). Hence \(Q \in \text{im}(f)\) and, to estimate \(\dim(f^{-1}(Q))\), consider the \((r - h - 1)\)-tuples of points \(P_{h+2}, \ldots, P_r\) such that \(\dim((Q, P_{h+2}, \ldots, P_r)) \leq r - h - 2\), i.e. the set \(Z(Q) := \{(P_{h+2}, \ldots, P_r) \in L_{h+2} \times L_{h+3} \times \cdots \times L_r| \dim((Q, P_{h+2}, \ldots, P_r)) \leq r - h - 2\}\). Note that, as \(r - (h + 1) \geq h + 2 \geq 3\), we have \(r \geq h + 4\). Hence in \(Z(Q)\) there are at least pairs of points.

Notice that, for the generic \((r - h - 1)\)-tuple \((P_{h+2}, \ldots, P_r) \in Z(Q)\), we have \(\dim((Q, P_{h+2}, \ldots, P_r)) = r - h - 2 = \dim((P_{h+2}, \ldots, P_r))\). Indeed the generic \((r - h - 1)\)-tuple \((P_{h+2}, \ldots, P_r) \in Z(Q)\) is a proper subset of a generic \(r\)-tuple of \(W_r\) and by Lemma 2.2, part 2), we have \(\dim((P_{h+2}, \ldots, P_r)) = r - h - 2\). On the other hand \(\dim((P_{h+2}, \ldots, P_r)) \leq \dim((Q, P_{h+2}, \ldots, P_r)) \leq r - h - 2\) by the definition of \(Z(Q)\). Then one can define a map \(\psi : Z \to W_{r-h-1}\), where \(Z\) is a non empty Zariski-open subset of \(Z(Q)\), by setting \(\psi(P_{h+2}, \ldots, P_r) = (P, P_{h+3}, \ldots, P_r)\), where \((P_{h+2}, \ldots, P_r)\) is a generic element of \(Z(Q)\) and \(P\) is the unique intersection, in \(\langle Q, P_{h+2}, \ldots, P_r \rangle = \langle P_{h+2}, \ldots, P_r \rangle\) of the line \(L_{h+2}\) with the linear subspace \((P_{h+3}, \ldots, P_r)\). Notice that \(\langle P_{h+3}, \ldots, P_r \rangle\) has codimension 1 in \(\langle Q, P_{h+2}, \ldots, P_r \rangle = \langle P_{h+2}, \ldots, P_r \rangle\) and it does not contain \(L_{h+2}\). Obviously \((P, P_{h+3}, \ldots, P_r) \in W_{r-h-1}\). The generic fibre of \(\psi\) is contained in \(L_{h+2}\) and therefore it has dimension 1 at most. It follows that \(\dim(Z(Q)) \leq \dim(W_{r-h-1}) + 1\). So we get: \(\dim(Z(Q)) \leq \dim(W_{r-h-1}) + 1 \leq r - 2h + 3 - 1 = r - 2h - 2\) by induction. Hence \(\dim(W_r) \leq 2h + r - N + \dim(f^{-1}(Q)) \leq 2h + r - N + \dim(Z(Q)) \leq 2h + r - N + r - 2h - 2 = 2r - N - 2 \leq r - 2\) and we are done. \(\square\)

### 3. Corollaries of the main theorem

In this section we give a list of 5 corollaries of Theorem 2.1. The first two corollaries contain our answer to the question in Section 1. The third one proves a property of the open Zariski set \(A_r\) which is defined in the previous corollaries. The last two show that, under the assumption \(r + 1 \leq N\), we can say more about the hyperplanes cutting \(P_1, \ldots, P_r\) on the lines of \(L\).

**Corollary 3.1.** With the same assumptions of Theorem 2.1 there exists a non empty, Zariski-open set \(A_r \subseteq L_1 \times L_2 \times \cdots \times L_r \simeq (\mathbb{P}^1)^{\times r}\) such that, for every \((P_1, \ldots, P_r) \in A_r\), it is \(\dim((P_1, \ldots, P_r)) = r - 1\), and the generic hyperplane of \(\mathbb{P}^N\) passing through \(P_1, \ldots, P_r\) does not contain any line of \(L\).

**Proof.** Let \(J_1, J_2, \ldots, J_r\) be the \(r\) varieties defined by removing, respectively, the first, the second, ..., the \(r\)th factor of \((\mathbb{P}^1)^{\times r}\). Let \(p_1, p_2, \ldots, p_r\) be the natural projections \(p_i : (\mathbb{P}^1)^{\times r} \to J_i\). By Theorem 2.1 we know that \(\dim(W_i) \leq r - 2\) in \((\mathbb{P}^1)^{\times r}\), hence \(p_i^{-1}(W_i)\) is a closed subscheme of dimension \(\leq r - 1\) in \((\mathbb{P}^1)^{\times r}\), for any \(i = 1, \ldots, r\). In \((\mathbb{P}^1)^{\times r}\), let \(A_r\) be the complement of the union of the \(r\) closed subschemes \(p_i^{-1}(W_i)\). Obviously \(A_r\) is a non empty Zariski-open set in \((\mathbb{P}^1)^{\times r}\) and \(\dim((P_1, \ldots, P_r)) = r - 1\) for every \(r\)-tuple \((P_1, \ldots, P_r) \in A_r\) because \((P_1, \ldots, P_r) \notin W_r\). Choose \(L_t \in L\) and, by contradiction, let us assume that every hyperplane in \(\mathbb{P}^N\) passing through \(P_1, \ldots, P_r\) contains \(L_t\). This would imply that there exists a point \(Q \in L_t\) (\(Q \neq P_t\)) such that \(\dim((P_1, \ldots, P_{t-1}, Q, P_{t+1}, \ldots, P_r)) = r - 2\) and therefore \((P_1, \ldots, P_{t-1}, Q, P_{t+1}, \ldots, P_r) \in W_r\). In fact: if all the hyperplanes passing through \(P_1, \ldots, P_r\) contain \(L_t\), this line belongs to \(\langle P_1, \ldots, P_r \rangle\), which is the intersection of all hyperplanes passing through \(P_1, \ldots, P_r\); in the
(r − 1)-dimensional linear space \( \langle P_1, \ldots, P_r \rangle \) there is the (r − 2)-dimensional 
subspace \( \langle P_1, P_2, P_3, \ldots, P_r \rangle \) and the line \( L_i \) cuts this subspace at a point Q. But \( (P_1, \ldots, P_r, Q) \) cannot belong to \( W_r \) because \( (P_1, \ldots, P_r, Q) \in P_i \) cannot belong to \( W_r \). Therefore, \( (P_1, \ldots, P_r, Q, P_i) \) \( \in W_r \) and if \( (P_1, \ldots, P_r, Q, P_i, \ldots, P_r) \in W_r \) the \( r \)-tuple \( (P_1, \ldots, P_r) \) would belong to the complement of \( A_r \).

**Corollary 3.2.** With the same assumptions of Theorem 2.1, there exists a non 
empty Zariski-open set \( H \subseteq \mathbb{P}^N \) whose points correspond to hyperplanes in \( \mathbb{P}^N \) 
cutting the set of lines \( L_1, \ldots, L_r \) only at an \( r \)-tuple of points \( P_1, \ldots, P_r \) with \( (P_1, \ldots, P_r) \in A_r \); moreover, for any non empty Zariski-open sets \( H' \subseteq H \) and \( A_r' \subseteq A_r \) and for any generic \( (P_1, \ldots, P_r) \in A_r \) there is at least a point in \( H' \) 
intersecting a hyperplane in \( \mathbb{P}^N \) cutting the set of lines \( L_1, \ldots, L_r \) only at the 
\( r \)-tuple of points \( P_1, \ldots, P_r \).

**Proof.** To prove Corollary 3.2, let us consider the incidence variety:

\[ I = \{ (H, P_1, \ldots, P_r) \in \mathbb{P}^N \times (\mathbb{P}^1)^r \mid P_1, \ldots, P_r \in H \} \]

and its natural projections \( \alpha : I \to \mathbb{P}^N \) and \( \beta : I \to (\mathbb{P}^1)^r \). Note that \( \alpha \) is 
both surjective and the dimension of the generic fibre of \( \alpha \) is zero because a generic 
hyperplane of \( \mathbb{P}^N \) intersects every line of \( L \) at one point only; thus \( \dim(I) = N \). 
For any fixed \( r \)-tuple of points \( (P_1, \ldots, P_r) \in A_r \), there exists a linear subspace 
\( \Lambda(P_1, \ldots, P_r) \) in \( \mathbb{P}^N \), given by the hyperplanes of \( \mathbb{P}^N \) passing through \( P_1, \ldots, P_r \); we 
have \( \dim(\Lambda(P_1, \ldots, P_r)) = N - r \), because \( P_1, \ldots, P_r \) are linearly independent. It 
follows that \( \dim(\beta^{-1}(A_r)) = r + N - r = N \) for the non empty Zariski-open subset 
\( \beta^{-1}(A_r) \subseteq I \), and therefore \( I = \beta^{-1}(A_r) \). Moreover, as every hyperplane either 
cuts every line \( L_1, \ldots, L_r \) at one point only or it contains the line entirely, the 
generic hyperplane of \( \Lambda(P_1, \ldots, P_r) \) contains only the fixed \( r \)-tuple. If it contains other 
\( r \)-tuples it will then contain at least one of the lines of \( L \) but this is not possible as 
\( (P_1, \ldots, P_r) \in A_r \).

The above discussion shows that a generic point of \( \beta^{-1}(A_r) \) can be represented 
as a pair \( \{ H, (P_1, \ldots, P_r) \} \), where \( H \) is a hyperplane cutting every \( L_1, \ldots, L_r \) only 
at the points \( P_1, \ldots, P_r \) with \( (P_1, \ldots, P_r) \in A_r \). Hence there exists a subset \( I' \subseteq \beta^{-1}(A_r) \) 
given by these pairs and \( I' \) is a non empty Zariski-open set of \( I \). To see 
this, for any \( i = 1, \ldots, r \), let \( C_i \) be the Zariski closed set in \( \mathbb{P}^N \), given by all 
hyperplanes containing \( L_i \). Every \( C_i \times (\mathbb{P}^1)^{r - i} \) is a closed set of \( \mathbb{P}^N \). 
Let \( T \) be the union of the union of these closed sets in \( \mathbb{P}^N \). Every \( \alpha(I') \) is a non empty Zariski-open set \( T \) with \( I' \), so that \( T = \beta^{-1}(A_r) \). 
Then \( \dim(\alpha(I')) = \dim(\alpha(I)) = N \) and therefore the interior of \( \alpha(I') \) is not empty. 
Letting \( \mathcal{H} \) be the interior of \( \alpha(I') \), one concludes the proof of the first part 
of Corollary 3.2. To prove the second part it suffices to change \( A_r \) with \( A_r' \) : the 
interior of \( \alpha(I') \) will intersect any non empty Zariski-open set \( \mathcal{H} \).

**Corollary 3.3.** With the same assumptions of Theorem 2.1, for every \( L_j \in \mathcal{L} \) there 
exists a finite subset of points \( K_j \subseteq L_j \) possibly empty, such that for every point 
\( P_j \in L_j \setminus K_j \), the intersection \( A_r \cap \{ P_j \} \) is an open, non empty, Zariski set of 
\( (\mathbb{P}^1)^{r - 1} \).

**Proof.** To prove Corollary 3.3 it is sufficient to remark that, as \( A_r \) is a non empty 
Zariski-open set in \( (\mathbb{P}^1)^r \), its projection onto any factor \( L_j \) is a non empty 
Zariski-open set in \( L_j \). This open set is the complement of a finite set \( K_j \) of 
points (possibly empty). For every point \( P_j \in L_j \setminus K_j \), \( A_r \) can not be contained in the
complement of the closed set $L_1 \times L_2 \times \ldots \{P_j\}, \cdot \cdot \cdot \times L_r$ and $A_r$ intersects this closed set along a non empty Zariski-open subset of it.

**Corollary 3.4.** Let us assume that $r + 1 \leq N$, and that there exist $r + 1$ lines $L_0, L_1, \ldots, L_r$ satisfying the assumptions of Theorem 2.1. Let $P$ be any point on $L_0$ and let $Z_P \subseteq \mathbb{P}^{N+1}$ be the dual hyperplane of $P$. Then there exists a non empty Zariski-open set $A_P \subseteq Z_P \sim \mathbb{P}^{N-1}$ such that every hyperplane in $\mathbb{P}^N$ corresponding to a point in $A_P$ cuts the lines $L_1, \ldots, L_r$ only at an $r$-tuple of points $P_1, \ldots, P_r$, with $(P_1, \ldots, P_r) \in A_r$.

**Proof.** Let us fix $P \in L_0$. By Theorem 2.1 applied to the $r + 1$ lines $L_0, L_1, \ldots, L_r$, we have $\dim(W_{r+1}) \leq r - 1$, hence $\dim(W_{r+1} \cap ([P] \times (\mathbb{P}^1)^{\times r} \simeq (\mathbb{P}^1)^{\times r})) \leq r - 1$. Therefore there exists a non empty Zariski-open set $B_P \subseteq (\mathbb{P}^1)^{\times r}$ such that $\dim((P, P_1, \ldots, P_r)) = r$ for every choice of $(P_1, \ldots, P_r) \in B_P$.

By Corollary 3.1 we know that there exists a non empty Zariski-open set $A_r$ in $(\mathbb{P}^1)^{\times r}$ such that $\dim((P_1, \ldots, P_r)) = r - 1$ for every choice of $(P_1, \ldots, P_r) \in A_r$ (and the generic hyperplane of $\mathbb{P}^N$ passing through $P_1, \ldots, P_r$ does not contain any line of $\mathcal{L}$). Let $C_P = B_P \cap A_r$. Then $C_P$ is a Zariski-open set in $(\mathbb{P}^1)^{\times r}$ such that $\dim((P_1, \ldots, P_r)) = r - 1$ and $\dim((P_1, \ldots, P_r)) = r$ for every choice of $(P_1, \ldots, P_r) \in C_P$ (and the generic hyperplane of $\mathbb{P}^N$ passing through $P_1, \ldots, P_r$ does not contain any line of $\mathcal{L}$).

Now, to prove Corollary 3.4, let us consider the incidence variety: $I = \{(H, P_1, \ldots, P_r) \in Z_P \times L_1 \times L_2 \times \cdots \times L_r \simeq \mathbb{P}^{N-1} \times (\mathbb{P}^1)^{\times r} | P_1, \ldots, P_r \in H\}$ and its natural projections $\alpha : I \rightarrow Z_P$ and $\beta : I \rightarrow (\mathbb{P}^1)^{\times r}$. As in the proof of Corollary 3.2, the dimension of the generic fibre of $\alpha$ is zero, because a generic hyperplane of $Z_P$ cuts every line $L_1, \ldots, L_r$ at one point only, hence $\dim(I) = N - 1$. For every $r$-tuple $(P_1, \ldots, P_r) \in C_P$, $\beta^{-1}(P_1, \ldots, P_r)$ is given by the hyperplanes of $Z_P$ passing through $P_1, \ldots, P_r$, i.e. by the hyperplanes of $\mathbb{P}^N$ passing through $P, P_1, \ldots, P_r$. As $\dim((P, P_1, \ldots, P_r)) = r$ we have that $\dim(\beta^{-1}(P_1, \ldots, P_r)) = N - (r + 1) \geq 0$. As $C_P$ is a non empty Zariski-open set of $(\mathbb{P}^1)^{\times r}$, $N - (r + 1)$ is also the dimension of the generic fibre of $\beta$ and therefore $\dim(\beta^{-1}(C_P)) = N - (r + 1) + r = N - 1$ and thus $I = \beta^{-1}(C_P)$. Then $\dim(\alpha(\beta^{-1}(C_P))) = N - 1$ and therefore its interior $U_0 \subseteq Z_P$ is not empty. Hence there exists a non empty Zariski-open set $U_0 \subseteq Z_P$ such that every point of $U_0$ corresponds to a hyperplane in $\mathbb{P}^N$ containing $P$ and an $r$-tuple of points $P_1, \ldots, P_r$ with $(P_1, \ldots, P_r) \in A_r$. On the other hand, for every line $L_i \in \mathcal{L}$, there exists a non empty Zariski-open set $U_i \subseteq Z_P$ given by the hyperplanes of $Z_P$ not containing $L_i$. Let $A_P = U_0 \cap U_1 \cap \cdots \cap U_r$. $A_P$ is a non empty Zariski-open set in $Z_P$ such that each one of its points corresponds to a hyperplane in $\mathbb{P}^N$ passing through $P$ and cutting the lines $L_1, \ldots, L_r$ at $r$ points $P_1, \ldots, P_r$ only, with $(P_1, \ldots, P_r) \in A_r$.

**Corollary 3.5.** With the same assumptions of Corollary 3.4, let $A'_r \subseteq A_r$ be any non empty Zariski-open subset. Then for every point $P \in L_0$, there exists a non empty Zariski-open set $A'_{P} \subseteq Z_P \sim \mathbb{P}^{N-1}$ such that every hyperplane in $\mathbb{P}^N$ corresponding to a point in $A'_{P}$ cuts the lines $L_1, \ldots, L_r$ only at an $r$-tuple of points $P_1, \ldots, P_r$, with $(P_1, \ldots, P_r) \in A'_r$; moreover, for any non empty Zariski-open sets $A'_{P} \subseteq Z_P$ and for any generic $(P_1, \ldots, P_r) \in A'_r$, there is at least a point in $A'_{P}$ corresponding to a hyperplane in $\mathbb{P}^N$ cutting the set of lines $L_1, \ldots, L_r$ only at the $r$-tuple of points $P_1, \ldots, P_r$. 

\[ \square \]
Proof. To prove the first part of Corollary 3.5 it suffices to change $A'_r \subseteq A_r$ with $A_r$ in the proof of Corollary 3.4. To prove the second part it suffices to intersect $A''_p$ with $A'_p$. \hfill \Box

4. Lines on rational scrolls

Let $S$ be a smooth, rational, scroll surface in $\mathbb{P}^N$ such that $S = \mathbb{P}(\mathcal{L})$, where $\mathcal{L} = \mathcal{O}_{\mathbb{P}^i}(h) \oplus \mathcal{O}_{\mathbb{P}^j}(k)$ with $1 \leq h \leq k$, $N = h + k + 1$, and $S$ is embedded in $\mathbb{P}^N$ by its tautological line bundle. Such scrolls are surfaces of minimal degree and projectively normal. By Lemma 2.1 we know that the assumptions of Theorem 2.1 are satisfied when $\mathcal{L} = \{L_1, \ldots, L_r\}$ is any set of $r$ lines in $\mathbb{P}^N$ which are fibres of a scroll such $S$, with $2 \leq r \leq N$. As usual $C_0$ and $f$ will be the numerical classes of the fundamental section and of any fibre of $S$, respectively. We have that $-C_0^2 = e = k - h$, where $e$ is the invariant of $S$ (see [2, V.2] for all references about ruled surfaces).

In this section we will always assume that $\mathcal{L} = \{L_1, \ldots, L_r\}$ is a set as above and $r \geq 3$. We will show that Theorem 2.1 can be made more precise for these sets of lines when $r \geq k + 2$ by using the existence of a well known incidence relation $I_r$, see below, however the theorem cannot be improved in this way.

First of all, let us recall that, by Lemma 2.2 1), to get any bound on the dimension on $W_r$, when $\mathcal{L} = \{L_1, \ldots, L_r\}$ is a set as above, we can assume that $\dim((P_1, \ldots, P_r)) = r - 2$ for any generic $(P_1, \ldots, P_r) \in W_r$. Hence let us consider the set $\widehat{W}_r := \{(P_1, \ldots, P_r) \in S^{(r)} \mid P_1, \ldots, P_r \text{ are distinct, belonging to } r \text{ distinct lines of } S \text{ and } \dim((P_1, \ldots, P_r)) = r - 2\}$. Because we can choose $r$ lines among the fibres of $S$ in $\infty^r$ ways, we have $\dim(\widehat{W}_r) = \dim(W_r) + r$. Hence, to get a bound on the dimension on $W_r$, when $\mathcal{L} = \{L_1, \ldots, L_r\}$ is a set as above, it suffices to get a bound for the dimension of $\widehat{W}_r$.

Let $G$ be the Grassmannian $G(r - 2, N)$ of the $(r - 2)$-dimensional linear spaces of $\mathbb{P}^N$, let $S^{(r)}$ be the $r$-symmetric product of $S$. We can consider the incidence variety $I_r \subseteq S^{(r)} \times G$ such that:

$$I_r := \{(P_1, \ldots, P_r, \Pi) \in S^{(r)} \times G \mid P_1, \ldots, P_r \in \Pi\} \quad (*)$$

with the two natural projections $p : I_r \to S^{(r)}$ and $q : I_r \to G$. Note that $\widehat{W}_r \subseteq \text{Im}(p)$, moreover the fibre of $p$ over $\widehat{W}_r$ is given by only one $(r - 2)$-dimensional linear space, so that $\dim(\widehat{W}_r) = \dim[p^{-1}(\widehat{W}_r)]$. Therefore to get bounds on the dimension of $\widehat{W}_r$ it is sufficient to get bounds on the dimension of $p^{-1}(\widehat{W}_r)$ by using $q$.

To investigate the fibre of the restriction of $q$ to $p^{-1}(\widehat{W}_r)$, let us put $\overline{W}_r := q[p^{-1}(\widehat{W}_r)]$ and let us consider the fibre over the generic $\Pi \in \overline{W}_r$. It is a linear space of dimension $r - 2$ cutting $S$ at $r$ distinct point belonging to $r$ distinct lines of $S$. The fibre of $q$ over $\Pi$ has positive dimension, for instance, when $\Pi$ contains a curve $\Gamma$ which is a smooth, irreducible section of $S$, and in this case the fibre has dimension $r$, because we could choose any set of $r$ points on $\Gamma$. About such a section $\Gamma$, we have the following

**Lemma 4.1.** Let $S$ be a surface as above. Let $\Gamma$ be a smooth, irreducible section of $S$, $\Gamma = C_0 + b$, such that $\dim(\langle \Gamma \rangle) = N - t$, $t \geq 1$. Then $b = k + 1 - t$ and $1 \leq t \leq h + 1$. 


Let \( H = C_0 + kf \) be the numerical class of the hyperplane section of \( S \). Obviously \( b \leq k \), and \( k - b \leq b \), as \( \Gamma \) is supposed to be a smooth, irreducible section of \( S \). Let us consider the exact sequence: \( 0 \to H \to \Gamma \to H|_\Gamma \to 0 \). As \( h^1(S, H - \Gamma) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-b)) = 0 \), we have:
\[
N + 1 = h + k + 2 = \dim(S, H) = h^0(S, H - \Gamma) + h^0(\Gamma, H|_\Gamma) = (k-b+1) + (N-t+1).
\]
Hence: \( b = k+1-t \) and it must be: \( k-h \leq k+1-t \leq k \), i.e. \( 1 \leq t \leq h+1 \).

Now let us return to the fibre over the generic \( \Pi \in \mathbb{W}_r \). By Lemma 4.1, if \( \Pi \) contains a section \( \Gamma \) as above, given that \( (\Gamma) \subseteq \Pi \), then \( N-t \leq r-2 \) with \( t \leq h+1 \), hence \( N+2-r \leq h+1 \), hence \( h+k+3-r \leq h+1 \), hence \( r \geq k+2 \). It follows that \( r \geq k+2 \) is exactly the range for which sections as \( \Gamma \) can occur.

Let \( V \) be the subvariety of \( G \) parametrizing \((r-2)\)-dimensional linear spaces of \( \mathbb{P}^N \) which are \((r-1)\)-secant \( S \). Obviously \( \dim(q) \subseteq V \), but \( \dim(q) \neq V \) and \( \dim(V) = 2(r-1) \), so that \( \dim(\dim(q)) < 2r-2 \). If \( \dim\{q[p^{-1}(\tilde{W}_r)]\} \), then \( \dim(\tilde{W}_r) = \dim[p^{-1}(\tilde{W}_r)] < 2r-2 \), hence \( \dim(W_r) < r-2 \) thus giving a stronger bound; but we saw above that fibres of the restriction of \( q \) to \( p^{-1}(\tilde{W}_r) \) can be of positive dimension when \( r \geq k+2 \), hence we cannot use \( I_r \) to improve Theorem 2.1. However we can prove the following

**Proposition 4.1.** Let \( S \) be a surface as above and let \( \tilde{W}_r \) be defined as above. Assume that \( N \geq r \geq k+2 \), then \( \dim(\tilde{W}_r) = 2r-2 \) and \( \dim(W_r) = r-2 \).

**Proof.** We know that \( \dim(\tilde{W}_r) = \dim(W_r) + r \), so we can consider only \( \tilde{W}_r \). By Theorem 2.1 it is sufficient to show that \( \dim(\tilde{W}_r) \geq 2r-2 \).

Let us put \( r = k+2+\eta \) with \( 0 \leq \eta \leq h-1 \). Utilizing again the incidence relation (*) introduced above, we will show that \( \dim(\tilde{W}_r) = \dim[p^{-1}(\tilde{W}_r)] \geq e+2\eta+1+r \) for any \( \eta \) with \( 0 \leq \eta \leq h-1 \). By choosing \( \eta = h-1 \) we will have \( \dim[p^{-1}(\tilde{W}_r)] \geq 2r-2 \).

Let us fix an \( e = e+\eta \) such that \( e = e+\eta \geq 1 \) distinct fibres on \( S \) and let us consider all hyperplanes in \( \mathbb{P}^N \) containing such fibres: \( h^0(S, H -(h-\eta)f) = h^0(S, C_0 + (e+\eta)f) = e+2\eta+2 \geq 2 \). This means that on \( S \) there exist a family of dimension at least \( e+2\eta+1 \) of curves \( \Gamma = C_0 + (e+\eta)f \), possibly reducible, such that \( \dim(\Gamma) = N - h^0(S, H - \Gamma) - N - h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(h-\eta)) = N - (h-\eta+1) = r-2 \). Note that in any \( \Gamma \) \( \mathbb{P}^r \) there are at most a finite number of curves as \( \Gamma \), otherwise \( S \) would be contained in a projective space of dimension \( r-2 < N \).

Now let us recall the incidence variety \( I_r \) by the previous remark we have that the subvariety \( V_r \subseteq \mathbb{W}_r \subseteq G \) parametrizing subspace as \( (\Gamma) \) has dimension at least \( e+2\eta+1 \), hence \( \dim(\tilde{W}_r) \geq e+2\eta+1 \), moreover the fibre of \( q \) over any point of \( V_r \) has dimension at least \( r \), hence \( \dim[p^{-1}(\tilde{W}_r)] \geq e+2\eta+1+r \).

5. A SIMPLE APPLICATION

To conclude the paper we give a simple application of Corollary 3.2. As mentioned in the introduction, this was the original situation that brought us to consider the problem addressed in this note.

**Proposition 5.1.** Let \( \{S_1, S_2\} \) be a pair of surfaces in \( \mathbb{P}^N \) as in Section 4. Assume that the intersection \( S_1 \cap S_2 \) in \( \mathbb{P}^N \) consists only of \( r \) common fibres \( L_1, \ldots, L_r \) and that, at a generic point \( P \in L_i \), the tangent planes to \( S_1 \) and \( S_2 \) at \( P \) are distinct.
Then, for any generic choice of $r$ points $P_1,...,P_r$, $P_i \in L_i$, there is a hyperplane of $\mathbb{P}^N$ intersecting transversally $S_1 \cap S_2$ only at $P_1,...,P_r$.

Proof. Apply Corollary 3.2 to $L := \{L_1,...,L_r\}$, keeping in mind that the assumptions of Theorem 2.1 are satisfied for any set of $r$ fibres on surfaces as above. □

Remark 5.1. Note that the set up of Proposition 5.1 is achieved, for instance, when every $S_j$ is $\mathbb{P}(E_{|\Gamma_j})$, where $E$ is a rank 2 vector bundle over a smooth variety $Y$, $\Gamma_1$ and $\Gamma_2$ are rational curves in $Y$ whose intersection is transverse and consists of $r$ distinct points, and $\mathbb{P}(E)$ is embedded in $\mathbb{P}^N$ as a scroll.

Following Remark 5.1, let $E$ be a rank 2 vector bundle over a smooth surface $Y$ which is rationally connected; let $X$ be $\mathbb{P}(E)$, let $T$ be its tautological divisor and let $\pi: X \to Y$ be the natural projection. In order to prove that the linear system $|T|$ separates two distinct points $P$ and $Q$ of $X$ you can consider a rational smooth curve $\Gamma$ (if it exists) passing through $\pi(P)$ and $\pi(Q)$, and the surface $S := \mathbb{P}(E_{|\Gamma})$. If $|T|_S$ is very ample and $|T| \to |T|_S$ is surjective then $|T|$ separates $P$ from $Q$. The difficult part of this strategy is often to prove the surjectivity (see for instance [1]).

The usual exact sequence $0 \to E \otimes O_Y(-\Gamma) \to E \to E_{|\Gamma} \to 0$ gives the required surjectivity if $h^1(Y,E \otimes O_Y(-\Gamma)) = 0$. Unfortunately, this vanishing is not always easy to control. One may choose a set $\{\Gamma = \Gamma_1,...,\Gamma_q\}$ of $q \gg 1$ suitable smooth rational curves in order to get $h^1(Y,E \otimes O_Y(-\Gamma_1...-\Gamma_q)) = 0$ and then use a reducible surface $S' := S_1 \cup ... \cup S_q$, instead of $S$, with $S_j := \mathbb{P}(E_{|\Gamma_j})$. With this approach one needs to consider elements of $|T|_{S'}$. Even when $|T|_{S_j}$ is very ample for any $j$, and $\Gamma_i \cap \Gamma_j$ is a set of distinct points for any $i,j$, to get sections of $|T|_{S'}$ it is crucial to know what elements of $|T|$ cut $S_i \cap S_j$ only at distinct points. Proposition 5.1 gives the answer.

References

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