POSITION VECTORS OF ADMISSIBLE CURVES IN 3-DIMENSIONAL PSEUDO-GALILEAN SPACE $G^3_1$

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Abstract. In this paper, position vectors of admissible curves in pseudo-Galilean space $G^3_1$ is studied in terms of Frenet equations. We compute the position vectors of admissible curves in pseudo-Galilean space $G^3_1$. Then we give some examples of position vectors for admissible curves.

1. Introduction

In the local differential geometry, curves are a geometric set of points, or locus. Intuitively, one can think a curve as the path traced out by a particle moving in Euclidean 3-space. So, to determine behaviour of the particle ( or the curve, i.e.) we investigate position vectors of curves.

In the Euclidean space $E^3$, for each unit speed curve $\alpha : I \rightarrow E^3$ with minimum four continuous derivatives, we can denote orthogonal unit vector fields $t, n$ and $b$ called, respectively, the tangent, the principal normal and the binormal vector fields. The planes spanned by $\{t, n\}, \{t, b\}$ and $\{n, b\}$ are called, respectively, osculating plane, rectifying plane and normal plane of the curve $\alpha$. If position vector of $\alpha : I \subset \mathbb{R} \rightarrow E^3$ always lie in its rectifying plane, the curves $\alpha$ are called rectifying curves. Similarly, the curves whose position vector $\alpha$ always lie in their osculating plane and their normal plane, are called, respectively, osculating curves and normal curves. In [3] B.Y. Chen expressed characterization of rectifying curve. Then, the characterization of rectifying curves in Minkowski space is given in [6].

In the Euclidean space $E^3$, the determination of parametric representation for position vector of arbitrary space with respect to intrinsic equations is still unknown [5,9]. Generally, to solve the above problem is difficult. But, the problem is solved some special case for example the event of a plane curve and a helix. Ali give some differential equation to solve the problem in the event of a general helix and slant helix in Minkowski 3-space [1,2]. Also, in Minkowski space position vectors of a spacelike W-curve is given in [8].

The aim of this study is to solve the problem for admissible curves in pseudo-Galilean 3-space $G^3_1$. First of all, we define the position vector of an admissible
curve according to the Frenet frame and then we obtain the position vector of an admissible curve according to standard frame in the way of curvature and torsion in pseudo-Galilean 3-space $G^1_3$.

2. Preliminaries

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries. As in [4], pseudo-Galilean inner product can be written as

$$\langle v_1, v_2 \rangle = \begin{cases} x_1 x_2, & \text{if } x_1 \neq 0 \lor x_2 \neq 0, \\ y_1 y_2 - z_1 z_2, & \text{if } x_1 = 0 \land x_2 = 0, \end{cases}$$

where $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$ and the pseudo-Galilean norm of the vector $v = (x, y, z)$ defined by

$$\|v\| = \begin{cases} x, & x \neq 0, \\ \sqrt{|y^2 - z^2|}, & x = 0. \end{cases}$$

A vector $v = (x, y, z)$ is in $G^1_3$ is said to be non-isotropic if $x \neq 0$, otherwise it is isotropic. All unit non-isotropic vectors are of the form $(1, y, z)$. There are four types of isotropic vectors: spacelike ($y^2 - z^2 > 0$), timelike ($y^2 - z^2 < 0$) and two types of lightlike ($y = \pm z$) vectors. A non-lightlike isotropic vector is unit vector if $y^2 - z^2 = \pm 1$.

In pseudo-Galilean space a curve is given by

$$\alpha : I \subseteq \mathbb{R} \rightarrow G^1_3, \quad \alpha (t) = (x(t), y(t), z(t))$$

where $I \subseteq \mathbb{R}$ and $x(t), y(t), z(t) \in C^3$. A curve $\alpha$ given above is called an admissible curve if $x(t) \neq 0$.

The curves in pseudo-Galilean space are characterized as follows:

**Type I.**

An admissible curve $\alpha : I \subseteq \mathbb{R} \rightarrow G^1_3$ can be parameterized by arc length $t = s$, given in coordinate form

$$\alpha (s) = (s, y(s), z(s)).$$

Its curvature $\kappa(s)$ and torsion $\tau(s)$ are defined by

$$\kappa(s) = \sqrt{|y^2 - z^2|},$$

$$\tau(s) = \frac{\det(\alpha(s), \alpha(s), \alpha(s))}{\kappa^2(s)}.$$

The associated trihedron is given by

$$t(s) = \alpha(s) = (1, y(s), z(s)),$$

$$n(s) = \frac{1}{\kappa(s)} \alpha(s) = \frac{1}{\kappa(s)}(0, y(s), z(s)),$$

$$b(s) = \frac{1}{\kappa(s)}(0, z(s), y(s)).$$

The vectors $t(s), n(s)$ and $b(s)$ are called the vectors of tangent, principal normal and binormal line of $\alpha$, respectively. The curve $\alpha$ given by (2.1) is timelike, if $n(s)$
is spacelike vector. For derivatives of tangent vector \( t(s) \), principal normal vector \( n(s) \) and binormal vector \( b(s) \), respectively, the following Frenet formulas hold:

\[
t(s) = \kappa(s)n(s),
\]
\[
n(s) = \tau(s)b(s),
\]
\[
b(s) = \tau(s)n(s).
\]

**Type II.**

An admissible curve \( \beta : I \subseteq \mathbb{R} \rightarrow G_3^1 \) is given by \( \beta(x) = (x, y(x), 0) \) and for this admissible curve, the curvature \( \kappa(x) \) and the torsion \( \tau(x) \) are defined by

\[
\kappa(x) = y(x),
\]
\[
\tau(x) = \frac{a_2(x)}{a_3(x)},
\]
where \( a(x) = (0, a_2(x), a_3(x)) \). The associated trihedron is given by

\[
t(x) = (1, y(x), 0),
\]
\[
n(x) = (0, a_2(x), a_3(x)),
\]
\[
b(x) = (0, a_3(x), a_2(x)).
\]

For tangent vector \( t(x) \), principal normal vector \( n(x) \) and binormal vector \( b(x) \), the following Frenet formulas hold

\[
t(x) = \kappa(x)(\cosh \phi(x)n(x) - \sinh \phi(x)b(x)),
\]
\[
n(x) = \tau(x)b(x),
\]
\[
b(x) = \tau(x)n(x).
\]
where \( \phi \) is the angle between \( a(x) \) and the plane \( z = 0 \). [4]

3. **Position vectors of admissible curves in pseudo-Galilean space \( G_3^1 \)**

In this section, we give the position vectors of admissible curves according to Frenet frame in pseudo-Galilean space \( G_3^1 \).

**Theorem 3.1.** Let \( \alpha(x) = (x, y(x), z(x)) \) be an admissible curve with curvature \( \kappa(x) \) and torsion \( \tau(x) \neq 0 \) in \( G_3^1 \). Then its position vector is given by

\[
(3.1) \quad \alpha(x) = (x + c_1)t(x) + \left[ c_2 - \frac{1}{2}(x + c_1)\kappa(x)e^{\tau(x)dx}dx \right] \left[ e^{-\tau(x)dx}(n(x) + b(x)) \right] + \left[ c_3 + \frac{1}{2}(x + c_1)\kappa(x)e^{-\tau(x)dx}dx \right] \left[ e^{\tau(x)dx}(b(x) - n(x)) \right]
\]
where \( c_1, c_2 \) and \( c_3 \) are arbitrary constants.

**Proof.** Let \( \alpha(x) = (x, y(x), z(x)) \) be an admissible curve in \( G_3^1 \). If \( \lambda(x) \), \( \mu(x) \) and \( \gamma(x) \) are differentiable functions of \( x \in I \subset \mathbb{R} \), then we can write the position vector of \( \alpha \) in the following form

\[
(3.2) \quad \alpha(x) = \lambda(x)t(x) + \mu(x)n(x) + \gamma(x)b(x).
\]

Differentiating the equation (3.2) with respect to \( x \) and considering the Frenet equations (2.4), we get

\[
\lambda(x) - 1 = 0,
\]
Then the solution for the above equation is written
\begin{equation}
\lambda(x)\kappa(x) + \mu(x) + \gamma(x)\tau(x) = 0,
\end{equation}
\begin{equation}
\mu(x)\tau(x) + \gamma(x) = 0.
\end{equation}

Using the first equation of (3.3), we find
\begin{equation}
\lambda(x) = x + c_1,
\end{equation}
where \(c_1\) is an arbitrary constant. We can consider the variable \(t = \tau(x)dx\). So, all functions of \(x\) will turn into the functions of \(t\). The dot is used to denote the derivation with respect to \(t\) (prime is used to denote the derivative with respect to \(x\)). We can write the third equation of (3.3) as follows
\begin{equation}
\gamma(t) - \gamma(t) = \frac{\lambda(t)\kappa(t)}{\tau(t)}.
\end{equation}

Then the solution for the above equation is written
\begin{equation}
\gamma(t) = \left(c_2 - \frac{1}{2} \frac{\lambda(t)\kappa(t)}{\tau(t)} e^{\gamma(t)} dt\right) e^{-t} + \left(c_3 + \frac{1}{2} \frac{\lambda(t)\kappa(t)}{\tau(t)} e^{-t} dt\right) e^t,
\end{equation}
where \(c_2\) and \(c_3\) are arbitrary constants. If we differentiate the equation (3.7) with respect to \(t\) and substituting this in the equation (3.5), we have
\begin{equation}
\mu(t) = \left(c_2 - \frac{1}{2} \frac{\lambda(t)\kappa(t)}{\tau(t)} e^{\gamma(t)} dt\right) e^{-t} - \left(c_3 + \frac{1}{2} \frac{\lambda(t)\kappa(t)}{\tau(t)} e^{-t} dt\right) e^t.
\end{equation}

So, the equations (3.7) and (3.8) can be written
\begin{equation}
\gamma(x) = \left(c_2 - \frac{1}{2} \left(x + c_1\right)\kappa(x) e^{\tau(x)} dx\right) e^{-\tau(x)} dx
+ \left(c_3 + \frac{1}{2} \left(x + c_1\right)\kappa(x) e^{-\tau(x)} dx\right) e^{\tau(x)} dx,
\end{equation}
\begin{equation}
\mu(x) = \left(c_2 - \frac{1}{2} \left(x + c_1\right)\kappa(x) e^{\tau(x)} dx\right) e^{-\tau(x)} dx
- \left(c_3 + \frac{1}{2} \left(x + c_1\right)\kappa(x) e^{-\tau(x)} dx\right) e^{\tau(x)} dx.
\end{equation}

If we use the equations (3.4), (3.9) and (3.10) in (3.2) we obtain equation (3.1). \(\square\)

**Theorem 3.2.** Let \(\beta(x) = (x, y(x), 0)\) be an admissible curve with constant \(\phi\) angle and constant torsion \(\tau(x)\) in \(G_1^3\). Then its position vector is given by
\begin{equation}
\beta(x) = \left(x + c_1\right)\lambda(x) + c_2 e^{-\tau(x)} n(x),
\end{equation}
where \(c_1\), \(c_2\) and \(c_3\) are arbitrary constants.

**Proof.** Let \(\beta(x) = (x, y(x), 0)\) be an admissible curve in \(G_1^3\). Then we write its position vector in the following form
\begin{equation}
\beta(x) = \lambda(x)\lambda(t) + \mu(x)n(x)
\end{equation}
where $\lambda(x)$ and $\mu(x)$ are differentiable functions of $x \in I \subset \mathbb{R}$. We can suppose $\tau, \phi$ are constants. If we differentiate the above equation with respect to $x$ and considering Frenet equations (2.7), we get

$$
\lambda(x) - 1 = 0,
$$

(3.13)

$$
\lambda(x) \kappa \cosh \phi + \mu(x) = 0,
$$

$$
-\lambda(x) \kappa \sinh \phi + \mu(x) \tau = 0.
$$

Using the first equation of (3.13), we find

(3.14)

$$
\lambda(x) = x + c_1,
$$

where $c_1$ is an arbitrary constant. If we use the second and third equation of (3.13), we have

(3.15)

$$
\mu'(x) + \tau \coth \phi \mu = 0.
$$

The general solution of these equation is

(3.16)

$$
\mu(x) = c_2 e^{-\tau \coth \phi x}.
$$

Substituting equations (3.14), (3.16) to (3.12), we obtain equation (3.11). \hfill \Box

4. Position vectors of admissible curves with respect to standard frame of $G^1_3$

**Theorem 4.1.** Let $\alpha(x) = (x, y(x), z(x))$ be an admissible curve with curvature $\kappa(x)$ and torsion $\tau(x)$ in the pseudo-Galilean space $G^1_3$.

i) if $\alpha$ is an admissible curve with spacelike normal, then the position vector of $\alpha$ is given

$$
\alpha(x) = (x, \int \kappa(x) \cosh \left( \int \tau(x) dx \right) dx, \left( \kappa(x) \sinh \left( \int \tau(x) dx \right) dx \right) dx).
$$

ii) if $\alpha$ is an admissible curve with timelike normal, then the position vector of $\alpha$ is given

$$
\alpha(x) = (x, \int \kappa(x) \sinh \left( \int \tau(x) dx \right) dx, \left( \kappa(x) \cosh \left( \int \tau(x) dx \right) dx \right) dx).
$$

**Proof.** If $\alpha(x)$ is an admissible curve in $G^1_3$, then from the second equation of (2.4) we obtain

$$
b(x) = \frac{1}{\tau} n(x).
$$

Using the third equation of (2.4) we have

$$
\left( \frac{1}{\tau} n(x) \right) - \tau(x) n(x) = 0.
$$

We can write the above equation by the form

$$
\frac{d^2 n}{dt^2} - n = 0,
$$

(4.3)

where $t = \int \tau(x) dx$. 

i) Let $\alpha$ be an admissible curve with spacelike normal. The principal normal vector can be written
\[ n = (0, \cosh \theta(t), \sinh \theta(t)) . \]
Considering the vector $n$ in the equation (4.3) we have
\[ \left( \theta^2(t) - 1 \right) \cosh \theta(t) + \dot{\theta}(t) \sinh \theta(t) = 0, \]
\[ \left( \theta^2(t) - 1 \right) \sinh \theta(t) + \dot{\theta}(t) \cosh \theta(t) = 0. \]
Using above equations we get
\[ \dot{\theta}(t) = \pm 1, \quad \ddot{\theta}(t) = 0, \]
and from above equation we have $\theta(t) = \pm t = \pm \int \tau(x)dx$. We can take the positive sign for $\theta(t)$. Then the principal normal vector can be written
\[ n(x) = \left( 0, \cosh \left( \int \tau(x)dx \right), \sinh \left( \int \tau(x)dx \right) \right). \]
Using the principal normal vector we have
\[ t(x) = \int \kappa(x) \left( 0, \cosh \left( \int \tau(x)dx \right), \sinh \left( \int \tau(x)dx \right) \right) + c, \]
where $c$ is a constant vector. We can take $c = (1, 0, 0)$ because of the first component of tangent vector and then
\[ t(x) = \left( 1, \int \kappa(x) \cosh \left( \int \tau(x)dx \right), \int \kappa(x) \sinh \left( \int \tau(x)dx \right) \right). \]
Using above equation we find
\[ \alpha(x) = \int \left( 1, \int \kappa(x) \cosh \left( \int \tau(x)dx \right), \int \kappa(x) \sinh \left( \int \tau(x)dx \right) \right) dx \]
So the equation (4.1) is obtained.

ii) Let $\alpha$ be an admissible curve with timelike normal. The principal normal vector can be written
\[ n = (0, \sinh \theta(t), \cosh \theta(t)) . \]
Considering $n$ in the equation (4.3) we obtain
\[ \left( \theta^2(t) - 1 \right) \sinh \left( \dot{\theta}(t) \right) + \theta(t) \cosh \left( \dot{\theta}(t) \right) = 0, \]
\[ \left( \theta^2(t) - 1 \right) \cosh \left( \dot{\theta}(t) \right) + \dot{\theta}(t) \sinh \left( \dot{\theta}(t) \right) = 0. \]
Using above equations we get
\[ \dot{\theta}(t) = \pm 1, \quad \ddot{\theta}(t) = 0, \]
and from above equation we have $\theta(t) = \pm t = \pm \int \tau(x)dx$. We can take the positive sign for $\theta(t)$. Then the principal normal vector can be written
\[ n(x) = \left( 0, \sinh \left( \int \tau(x)dx \right) dx, \cosh \left( \int \tau(x)dx \right) dx \right). \]
Using above equation we have
\[ t(x) = \int \kappa(x) \left( 0, \sinh \left( \int \tau(x) \, dx \right), \cosh \left( \int \tau(x) \, dx \right) \right) + c, \]
where \( c \) is a constant vector. We can take \( c = (1, 0, 0) \) because of the first component of tangent vector and then
\[ t(x) = \left( 1, \int \kappa(x) \sinh \left( \int \tau(x) \, dx \right) \, dx, \int \kappa(x) \cosh \left( \int \tau(x) \, dx \right) \, dx \right). \]
Using above equation we obtain
\[ \alpha(x) = \int \left( 1, \int \kappa(x) \sinh \left( \int \tau(x) \, dx \right) \, dx, \int \kappa(x) \cosh \left( \int \tau(x) \, dx \right) \, dx \right) \, dx. \]

**Theorem 4.2.** Let \( \beta(x) = (x, y(x), 0) \) be an admissible curve with curvature \( \kappa(x) \) and torsion \( \tau(x) \) in the pseudo-Galilean space \( G^1_3 \).

i) if \( \beta \) be an admissible curve with spacelike normal, then the position vector of \( \beta \) is given
\[
\beta(x) = \left( x, \int \left( \int \kappa(x) \left( \cosh \phi \cosh \left( \int \tau(x) \, dx \right) - \sinh \phi \sinh \left( \int \tau(x) \, dx \right) \right) \, dx \right) \, dx, \right.
\]
(4.4) \[
\beta(x) = \left( x, \int \left( \int \kappa(x) \left( \cosh \phi \sinh \left( \int \tau(x) \, dx \right) - \sinh \phi \cosh \left( \int \tau(x) \, dx \right) \right) \, dx \right) \, dx, \right.
\]
(4.5)

ii) if \( \beta \) be an admissible curve with timelike normal, then the position vector of \( \beta \) is given
\[
\beta(x) = \left( x, \int \left( \int \kappa(x) \left( \cosh \phi \sinh \left( \int \tau(x) \, dx \right) - \sinh \phi \cosh \left( \int \tau(x) \, dx \right) \right) \, dx \right) \, dx, \right.
\]
(4.6)

Proof. i) Let \( \beta \) be an admissible curve with spacelike normal. If \( \beta(x) \) is an admissible curve in \( G^1_3 \), then the Frenet equations (2.7) are held. From the second equation of (2.7), we have
\[
b(x) = \frac{1}{\tau} n(x).
\]
Using the third equation of (2.7), we have
\[
\left( \frac{1}{\tau} n(x) \right) - \tau(x) n(x) = 0.
\]
So the above equation can be written
\[
\frac{d^2 n}{dt^2} - n = 0,
\]
(4.6)
where \( t = \int \tau(x) \, dx \). The principal normal vector can be written as follows
\[
n = (0, \cosh \theta(t), \sinh \theta(t)).
\]
If we use the vector $n$ in the equation (4.6) we obtain
\[
\begin{align*}
\left( \theta^2 (t) - 1 \right) \cosh \theta (t) + \dot{\theta}(t) \sinh \theta (t) &= 0, \\
\left( \theta^2 (t) - 1 \right) \sinh \theta (t) + \dot{\theta}(t) \cosh \theta (t) &= 0.
\end{align*}
\]
Then we get
\[
\dot{\theta}(t) = \pm 1 , \quad \ddot{\theta}(t) = 0,
\]
and from above equation we have $\dot{\theta}(t) = \pm t = \pm \int \tau(x)dx$. We can take the positive sign for $\dot{\theta}(t)$.

Then we have $n(x) = (0, \cosh \left( \int \tau(x)dx \right) dx, \sinh \left( \int \tau(x)dx \right) dx)$. We can take the positive sign for $\dot{\theta}(t)$.

Since $\beta(x)$ is an admissible curve in $G_3$, the Frenet equations (2.7) are hold. From the third equation (2.7), we have
\[
n(x) = \frac{1}{\tau} b(x).
\]
If we put the above equation in the second equation of (2.7) we obtain the differential equation with respect to principal normal vector $n$
\[
\left( \frac{1}{\tau} b(x) \right) - \tau(x) b(x) = 0.
\]

The above equation can be written as follows
\[(4.7) \quad \frac{d^2 b}{dt^2} - b = 0,
\]
where $t = \int \tau(x)dx$. We can write the binormal vector in the following form
\[b = (0, \sinh \theta (t) , \cosh \theta (t)) .
\]

Considering the second and the third components from the vector $n$ in the equation (4.7) we obtain
\[
\begin{align*}
\left( \theta^2 (t) - 1 \right) \sinh \theta (t) + \dot{\theta}(t) \cosh \theta (t) &= 0, \\
\left( \theta^2 (t) - 1 \right) \cosh \theta (t) + \dot{\theta}(t) \sinh \theta (t) &= 0.
\end{align*}
\]

So, using the above equations we get
\[
\dot{\theta}(t) = \pm 1 , \quad \ddot{\theta}(t) = 0,
\]
and from above equation we have $\dot{\theta}(t) = \pm t = \pm \int \tau(x)dx$. We can take the positive sign for $\dot{\theta}(t)$.

Then the principal normal vector is written as follows
\[b(x) = \left( 0, \sinh \left( \int \tau(x)dx \right) dx, \cosh \left( \int \tau(x)dx \right) dx \right) .
\]
Using first equation of (2.7), we can write
\[ t(x) = \kappa(x) \cosh \phi \left( 0, \cosh \left( \int \tau(x) dx \right), \sinh \left( \int \tau(x) dx \right) \right) \]
\[ -\kappa(x) \sinh \phi \left( 0, \sinh \left( \int \tau(x) dx \right), \cosh \left( \int \tau(x) dx \right) \right). \]

If we integrate the above equation with respect to \( x \), we have the equation (4.4).

ii) Let \( \beta \) be an admissible curve with timelike normal. If \( \beta(x) \) is an admissible curve in \( G_3^1 \), then the Frenet equations (2.7) are hold. From the second equation of (2.7), we obtain
\[ b(x) = \frac{1}{\tau} n(x). \]

Considering the above equation to the third equation of (2.7) we obtain
\[ \left( \frac{1}{\tau} n(x) \right) - \tau(x) n(x) = 0. \]

We can write the above equation in the following form
(4.8) \[ \frac{d^2 n}{dt^2} - n = 0, \]
where \( t = \int \tau(x) dx \). The principal normal vector can be written
\[ n = (0, \sinh \theta(t), \cosh \theta(t)). \]

Using the equation (4.8) we have
\[ \left( \dot{\theta}^2(t) - 1 \right) \sinh \theta(t) + \dot{\theta}(t) \cosh \theta(t) = 0, \]
\[ \left( \ddot{\theta}(t) - 1 \right) \cosh \theta(t) + \dot{\theta}(t) \sinh \theta(t) = 0. \]

So,
\[ \dot{\theta}(t) = \pm 1, \quad \ddot{\theta}(t) = 0, \]
and from above equation we have \( \dot{\theta}(t) = \pm t = \pm \int \tau(x) dx \). We can take the positive sign for \( \dot{\theta}(t) \). Then
\[ n(x) = \left( 0, \sinh \left( \int \tau(x) dx \right), \cosh \left( \int \tau(x) dx \right) \right). \]

Since \( \beta(x) \) is an admissible curve in \( G_3^1 \). From the third equation (2.7), we have
\[ n(x) = \frac{1}{\tau} b(x). \]

Considering the second equation of (2.7) we have
\[ \left( \frac{1}{\tau} b(x) \right) - \tau(x) b(x) = 0. \]

The above equation can be written
(4.9) \[ \frac{d^2 b}{dt^2} - b = 0, \]
where \( t = \int \tau(x) dx \). Thus
\[ b = (0, \cosh \theta(t), \sinh \theta(t)). \]
Using the equation (4.9) we have
\[
\begin{align*}
\left( \theta^2(t) - 1 \right) \cosh \theta(t) + \theta(t) \sinh \theta(t) & = 0, \\
\left( \theta^2(t) - 1 \right) \sinh \theta(t) + \theta(t) \cosh \theta(t) & = 0.
\end{align*}
\]
Then
\[\dot{\theta}(t) = \pm 1, \quad \ddot{\theta}(t) = 0,\]
and from above equation we have \( \theta(t) = \pm t = \pm \int \tau(x) dx \). We can take the positive sign for \( \theta(t) \). Then
\[b(x) = \left( 0, \cosh \left( \int \tau(x) dx \right), \sinh \left( \int \tau(x) dx \right) \right).\]

Using first equation of (2.7), we can write
\[t(x) = \kappa(x) \cosh \phi \left( 0, \sinh \left( \int \tau(x) dx \right), \cosh \left( \int \tau(x) dx \right) \right) - \kappa(x) \sinh \phi \left( 0, \cosh \left( \int \tau(x) dx \right), \sinh \left( \int \tau(x) dx \right) \right).\]
If we integrate the above equation with respect to \( x \), we get the equation (4.5).

**Example 4.1.** Let \( \alpha \) be a straight line with respect to the Frenet frame in \( G^1_3 \). If we take \( \kappa(x) = 0 \) and consider this in the equation (4.1) and (4.4), then its position vector can be written
\[\alpha_1(x) = (x, c_1 x + c_2, c_3 x + c_4)\]
and
\[\alpha_1(x) = (x, c_1 x - c_2 x + c_3, c_4 x + c_5 x + c_6),\]
respectively, where \( c_i, i = 1, 2, 3, 4, 5, 6 \) are arbitrary constants.

**Example 4.2.** Let \( \beta \) be a planar curve with respect to the Frenet frame in \( G^1_3 \). If we take \( \tau(x) = 0 \) and consider this in the equation (4.1) and (4.2), then its position vector can be written
\[\beta_3(x) = \left( x, \cosh \eta \int \left( \int \kappa(x) dx \right) dx, \sinh \eta \int \left( \int \kappa(x) dx \right) dx \right)\]
and
\[\beta_4(x) = \left( x, \sinh \eta \int \left( \int \kappa(x) dx \right) dx, \cosh \eta \int \left( \int \kappa(x) dx \right) dx \right),\]
respectively, where \( \eta \) is arbitrary constant. If we take \( \tau(x) = 0 \) and consider this in the equation (4.4) and (4.5), then its position vector can be written
\[\beta_5(x) = \left( x, \nu \int \left( \int \kappa(x) \cosh \phi(x) dx \right) dx - \delta \int \left( \int \kappa(x) \sinh \phi(x) dx \right) dx, \right.\]
\[\left. \delta \int \left( \int \kappa(x) \cosh \phi(x) dx \right) dx - \nu \int \left( \int \kappa(x) \sinh \phi(x) dx \right) dx \right)\]
and

\[ \beta_6(x) = \left( x, \delta \int \left( \int \kappa(x) \cosh \phi(x)dx \right) dx - \nu \int \left( \int \kappa(x) \sinh \phi(x)dx \right) dx, \right. \]
\[ \left. \nu \int \left( \int \kappa(x) \cosh \phi(x)dx \right) dx - \delta \int \left( \int \kappa(x) \sinh \phi(x)dx \right) dx \right) \]

respectively, where \( \eta, \nu \) and \( \delta \) are arbitrary constants, \( \cosh \eta = \nu \) and \( \sinh \eta = \delta \).

**Example 4.3.** Let \( \gamma \) be an admissible curve with \( \kappa(x) = \text{const.} \) and \( \tau(x) = \text{const.} \) in pseudo-Galilean space \( G^3 \). If we take \( \kappa(x) \) and \( \tau(x) \) are constants and put it in the equation (4.1) and (4.2), we obtain

\[ \gamma_1 = (x, \frac{\kappa}{\tau^2} \cosh (\tau x), \frac{\kappa}{\tau^2} \sinh (\tau x)) \]

and

\[ \gamma_2 = (x, \frac{\kappa}{\tau^2} \sinh (\tau x), \frac{\kappa}{\tau^2} \cosh (\tau x)) \]

respectively.

If we take \( \kappa(x) \) and \( \tau(x) \) are constants and consider this in the equation (4.4) and (4.5), we get

\[ \gamma_3 = (x, \int \left( \int \kappa(x) (\cosh \phi c \cosh \left[ \xi(x) \right] - \sinh \theta \sinh \left[ \xi(x) \right]) dx \right) dx, \]
\[ \int \left( \int \kappa(x) (\cosh \phi \sinh \left[ \xi(x) \right] - \sinh \theta \cosh \left[ \xi(x) \right]) dx \right) dx \]

**Example 4.4.** Let \( \psi \) be an admissible general helix in pseudo-Galilean space \( G^3 \). Then if we take \( \tau(x) = m \kappa(x) \), where \( m \) is arbitrary constant and consider this in the equation (4.1) and (4.2), we get

\[ \psi_1 = (x, \int \left[ \int \kappa(x) \left( \cosh \left( m \int \kappa(x)dx \right) \right) \left( \sinh \left( m \int \kappa(x)dx \right) \right) dx \right] dx) \]

and

\[ \psi_2 = (x, \int \left[ \int \kappa(x) \left( \sinh \left( m \int \kappa(x)dx \right) \right) \left( \cosh \left( m \int \kappa(x)dx \right) \right) dx \right] dx) \]

respectively.

If we take \( \tau(x) = m \kappa(x) \), where \( m \) is arbitrary constant and consider this in the equation (4.4) and (4.5), we get

\[ \psi_3 = (x, \int \left[ \int \kappa(x) \left( \cosh \phi \cosh \left[ \xi(x) \right] - \sinh \theta \sinh \left[ \xi(x) \right] \right) dx \right] dx, \]
\[ \int \left[ \int \kappa(x) \left( \cosh \phi \sinh \left[ \xi(x) \right] - \sinh \theta \cosh \left[ \xi(x) \right] \right) dx \right] dx) \]
and
\[
\psi_4 = (x, \int \left[ \int \kappa(x) \left( \cosh \phi \sinh [\xi(x)] - \sinh \theta \cosh [\xi(x)] \right) dx \right] dx,
\]
\[
\int \left[ \int \kappa(x) \left( \cosh \phi \cosh [\xi(x)] - \sinh \theta \sinh [\xi(x)] \right) dx \right] dx),
\]
respectively, where \( m \int \kappa(x) dx = \xi(x) \).

References