ON KILLING VECTOR FIELDS ON A TANGENT BUNDLE WITH $g$− NATURAL METRIC.

PART II

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Abstract. The tangent bundle of a Riemannian manifold $(M, g)$ with non-degenerate $g$− natural metric $G$ that admits a Killing vector field decomposes into four classes. Properties of these classes are investigated. A complete structure of the Lie algebra of Killing vector fields for some subclasses is given.

1. Introduction

In the first part of the paper ([9], see also [10]) we have developed the method by Tanno ([18]) to investigate Killing vector fields on $TM$ with an arbitrary, non-degenerate $g$− natural metric. The method applied Taylor’s formula to components of the vector field that was supposed to be an infinitesimal isometry. It is known that an infinitesimal affine transformation, in particular an infinitesimal isometry, is determined by the values of its components and their first partial derivatives at a point ([14], p. 232). It appears by applying the Taylor’s formula there are at most four generators of the infinitesimal isometry: two vectors and two tensors of type $(1, 1)$.

We have proved the following

Theorem 1.1. ([9], [10]) Let $(TM, G)$ be a tangent bundle of a Riemannian manifold $(M, g)$, $\dim M > 2$, with $g$− natural non-degenerate metric $G$. Let $Z$ be a Killing vector field on $TM$ with its Taylor series expansion around a point $(x, 0) \in TM$ given by (3.2) and (3.3). Then for each such a point there exists a neighbourhood $U \subset M, x \in U$, that one of the following cases occurs:

(1) $2ba_2 - a_1b_2 \neq 0$. Then

\begin{align*}
\nabla_k X_l + \nabla_l X_k &= 0, & \nabla_k Y_l + \nabla_l Y_k &= 0, \\
(1.1) & P_{kl} + P_{lk} &= 0, & K_{kl} + K_{lk} &= 0. (1.2)
\end{align*}
The Ricci identity is
\[ \nabla R = \nabla_k R_{ij}^l = \Gamma^s_{kl} R_{sj}^i + \Gamma^s_{jl} R_{sk}^i - \Gamma^s_{ij} R_{sk}^l - \Gamma^s_{ij} R_{sk}^l, \]

where \( \nabla \) denotes the Levi-Civita connection. We have
\[ \partial_t g_{kk} = g_{kk,t} = \Gamma_{kt}^r g_{rk} + \Gamma_{kt}^r g_{rkh} \]

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and either
\[ a_j = a_j(r^2)_{(x,0) \in TM}, b_j = b_j(r^2)_{(x,0) \in TM}, a_j^2 = a_j^2(r^2)_{(x,0) \in TM}, A = a_1 + a_3 \quad \text{and} \quad b = b_1 - a_1^2. \]

Above theorem splits \((TM, G)\) into four classes. In section 4 of the paper for each such class further properties are proved separately. Some restrictions on a number of generators are found (cf. for example 3.5 and Corollary after it). Moreover, a complete structure of Killing vector fields on \( TM \) for some subclasses is given (Theorems 4.3 and 4.7). In the next section some classical lifts of some tensor fields from \((M, g)\) to \((TM, G)\) are discussed.

Finally, in the Appendix we collect some known facts and theorems that we use throughout the paper.

Throughout the paper all manifolds under consideration are smooth and Hausdorff ones. The metric \( g \) of the base manifold \( M \) is always assumed to be Riemannian one.

The computations in local coordinates were partially carried out and checked using MathTensorTM and Mathematica software.

2. Preliminaries

2.1. Conventions and basic formulas. Let \((M, g)\) be a pseudo-Riemannian manifold of dimension \( n \) with metric \( g \). The Riemann curvature tensor \( R \) is defined by
\[ R(X, Y) = \nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X, Y]} Y. \]

In a local coordinate neighbourhood \((U, (x^1, \ldots, x^n))\) its components are given by
\[ R(\partial_i, \partial_j) \partial_k = R(\partial_i, \partial_j, \partial_k) = R_{ijk}^l \partial_l = (\partial_t \Gamma^r_{jk} - \partial_j \Gamma^r_{tk} + \Gamma^r_{js} \Gamma^s_{jk} - \Gamma^r_{js} \Gamma^s_{jk}) \partial_r, \]

where \( \partial_t = \frac{\partial}{\partial x^t} \) and \( \Gamma^r_{jk} \) are the Christoffel symbols of the Levi-Civita connection \( \nabla \). We have
\[ \partial_t g_{kk} = g_{kk,t} = \Gamma^r_{kt} g_{rk} + \Gamma^r_{kt} g_{rkh}. \]

The Ricci identity is
\[ \nabla_k R_{ij}^l = \nabla_j R_{ki}^l = \nabla_{ij} R_{kl}^l = 0, \]

and either
\[ a_j = a_j(r^2)_{(x,0) \in TM}, b_j = b_j(r^2)_{(x,0) \in TM}, a_j^2 = a_j^2(r^2)_{(x,0) \in TM}, A = a_1 + a_3 \quad \text{and} \quad b = b_1 - a_1^2. \]

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for all vector fields \( X, Y, Z \) on \( M \). In local coordinates \((U, (x^1, \ldots, x^n))\) we get
\[
(L_{X^r} \partial_i, g)_{ij} = \nabla_i X_j + \nabla_j X_i,
\]
where \( X_k = g_{kr} X^r \).

We shall need the following properties of the Lie derivative
\[
(2.2) \quad L_X \Gamma^h_{ji} = \nabla_j \nabla_i X^h + X^r R_{rjirs} g^{sh} = \frac{1}{2} g^{hr} [\nabla_j (L_X g_{ir}) + \nabla_i (L_X g_{jr}) - \nabla_r (L_X g_{ji})].
\]
If \( L_X \Gamma^h_{ji} = 0 \), then \( X \) is said to be an infinitesimal affine transformation.

The vector field \( X \) is said to be the Killing vector field or infinitesimal isometry if \( L_X g = 0 \), ([20], p. 23 and 24).

2.2. Tangent bundle. Let \( x \) be a point of a Riemannian manifold \((M, g)\), \( \dim M = n \), covered by coordinate neighbourhoods \((U, (x^j))\), \( j = 1, \ldots, n \). Let \( TM \) be tangent bundle of \( M \) and \( \pi : TM \to M \) be a natural projection on \( M \). If \( x \in U \) and \( u = u^r \frac{\partial}{\partial x^r} \in T_x M \), then \((\pi^{-1}(U), ((x^r), (u^r)))\), \( r = 1, \ldots, n \), is a coordinate neighbourhood on \( TM \).

For all \((x, u) \in TM\) we denote by \( V_{(x,u)} TM \) the kernel of the differential at \((x, u)\) of the projection \( \pi : TM \to M \), i.e.,
\[
V_{(x,u)} TM = \text{Ker} \left( d\pi_{(x,u)} \right),
\]
which is called the vertical subspace of \( T_{(x,u)} TM \) at \((x, u)\).

To define the horizontal subspace of \( T_{(x,u)} TM \) at \((x, u)\), let \( V \subset M \) and \( W \subset T_x M \) be open neighbourhoods of \( x \) and \( 0 \) respectively, diffeomorphic under exponential mapping \( \text{exp}_x : T_x M \to M \). Furthermore, let \( S : \pi^{-1}(V) \to T_x M \) be a smooth mapping that translates every vector \( Z \in \pi^{-1}(V) \) from the point \( y \) to the point \( x \) in a parallel manner along the unique geodesic connecting \( y \) and \( x \). Finally, for a given \( u \in T_x M \), let \( R_u : T_x M \to T_x M \) be a translation by \( u \), i.e. \( R_u(X_x) = X_x - u \). The connection map
\[
K_{(x,u)} : T_{(x,u)} TM \to T_x M
\]
of the Levi-Civita connection \( \nabla \) is given by
\[
K_{(x,u)}(Z) = d(\text{exp}_p \circ R_{-u} \circ S)(Z)
\]
for any \( Z \in T_{(x,u)} TM \).

For any smooth vector field \( Z : M \to TM \) and \( X_x \in T_x M \) we have
\[
K(dZ_x(X_x)) = (\nabla_X Z)_x.
\]
Then \( H_{(x,u)} TM = \text{Ker}(K_{(x,u)}) \) is called the horizontal subspace of \( T_{(x,u)} TM \) at \((x, u)\).

The space \( T_{(x,u)} TM \) tangent to \( TM \) at \((x, u)\) splits into direct sum
\[
T_{(x,u)} TM = H_{(x,u)} TM \oplus V_{(x,u)} TM.
\]
We have isomorphisms
\[
H_{(x,u)} TM \sim T_x M \sim V_{(x,u)} TM.
\]

For any vector \( X \in T_x M \) there exist the unique vectors: \( X^h \) given by \( d\pi(X^h) = X \) and \( X^v \) given for any function \( f \) on \( M \) by \( X^v(df) = X f \). The vectors \( X^h \) and...
Lemma 2.1. Let \( g \) be a natural metric on \( M \). Between them a class of so-called \( g \)-natural metrics is of special interest. The well-known Cheeger-Gromoll and Sasaki metrics are special cases of the \( g \)-natural metrics ([15]).

**Lemma 2.1.** ([4], [5]) Let \((M, g)\) be a Riemannian manifold and \( G \) be a \( g \)-natural metric on \( TM \). There exist functions \( a_j, b_j : (0, \infty) \rightarrow R, j = 1, 2, 3, \) such that for every \( X, Y, u \in T_u M \)

\[
G(x,u)(X^h,Y^h) = (a_1 + a_3)(r^2)g_x(X,Y) + (b_1 + b_3)(r^2)g_x(X,u)g_x(Y,u),
\]

\( (2.3) \) \[ G(x,u)(X^h,Y^v) = a_2(r^2)g_x(X,Y) + b_2(r^2)g_x(X,u)g_x(Y,u), \]

\[ G(x,u)(X^v,Y^h) = a_2(r^2)g_x(X,Y) + b_2(r^2)g_x(X,u)g_x(Y,u), \]

\[ G(x,u)(X^v,Y^v) = a_1(r^2)g_x(X,Y) + b_1(r^2)g_x(X,u)g_x(Y,u), \]

where \( r^2 = g_x(u,u) \). For \( \dim M = 1 \) the same holds for \( b_j = 0, j = 1, 2, 3. \)

Following ([4]) we put

1. \( a(t) = a_1(t)(a_1(t) + a_3(t)) - a_2^2(t), \)
2. \( F_j(t) = a_j(t) + b_j(t), \)
3. \( F(t) = F_1(t)[F_1(t) + F_2(t)] - F_2^2(t) \)

for all \( t \in (0, \infty) \).

We shall often abbreviate: \( A = a_1 + a_3, B = b_1 + b_3. \)

**Lemma 2.2.** ([4], Proposition 2.7) The necessary and sufficient conditions for a \( g \)-natural metric \( G \) on the tangent bundle of a Riemannian manifold \((M, g)\) to be non-degenerate are \( a(t) \neq 0 \) and \( F(t) \neq 0 \) for all \( t \in (0, \infty) \). If \( \dim M = 1 \) this is equivalent to \( a(t) \neq 0 \) for all \( t \in (0, \infty) \).

For a general overview on \( g \)-natural metric we refer the reader to ([1]), ([2]). The components of the Levi-Civita connection of an arbitrary, non-degenerate \( g \)-natural metric \( G \) are calculated in ([7]). They are the same as in the Riemannian case ([1], p. 112-113).

3. **Taylor’s formula for Killing vector field and coefficients**

Suppose now that

\[
Z = Z^a\partial_a + \tilde{Z}^a\delta_a = Z^a\partial_a^h + (\tilde{Z}^a + Z^a u\Gamma^a)\partial_a^v = H^a\partial_a^h + V^a\partial_a^v
\]
is a vector field on $TM$. Throughout the paper the following hypothesis will be used:

(3.1) $(M, g)$ is a Riemannian manifold of dimension $n$ with metric $g$, $H$ covered by the coordinate system $(U, (x^r))$.

($TM, G$) is the tangent bundle of $M$ with $g$ - natural non-degenerate metric $G$, covered by a coordinate system $(\pi^{-1}(U), (x^r, u^s))$, $r, s$ run through the range $\{1, ..., n\}$.

$Z$ is a Killing vector field on $TM$ with local components $(Z^r, \tilde{Z}^s)$ with respect to the local base $(\partial_r, \delta_s)$.

Let

(3.2) $H^a = Z^a = Z^a(x, u) = X^a + K^a_p u^p + \frac{1}{2} E^a_{pq} u^pu^q + \frac{1}{3!} G^a_{pqr} u^pu^qu^r + \frac{1}{4!} H^a_{pqrs} u^pu^qu^ru^s + \cdots$,

(3.3) $\tilde{Z}^a = \tilde{Z}^a(x, u) = Y^a + \tilde{P}^a_p u^p + \frac{1}{2} Q^a_{pq} u^pu^q + \frac{1}{3!} S^a_{pqrs} u^pu^qu^ru^s + \frac{1}{4!} V^a_{pqrs} u^pu^qu^ru^s + \cdots$

be expansions of the components $Z^a$ and $\tilde{Z}^a$ by Taylor’s formula in a neighbourhood of a point $(x, 0) \in TM$. For each index $a$ the coefficients are values of partial derivatives of $Z^a$ and $\tilde{Z}^a$ respectively, taken at a point $(x, 0)$ and therefore are symmetric in all lower indices. For simplicity we have omitted the remainders.

**Lemma 3.1.** ([18]) The quantities

\[ X = (X^a(x)) = (Z^a(x, 0)) \quad Y = (Y^a(x)) = (\tilde{Z}^a(x, 0)) \quad K = (K^a_p(x)) = (\delta_p Z^a(x, 0)) \quad E = (E^a_{pq}(x)) = (\delta_p \delta_q Z^a(x, 0)) \quad P = (P^a_p(x)) = (\delta_p \tilde{Z}^a(x, 0) - \partial_p (Z^a(x, 0))) \]

are tensor fields $M$.

We shall often use the following definitions and abbreviations:

\[ S^a_p = P^a_p + \nabla_p X^a, \quad S_{kp} = S^a_p g_{ak}, \quad P_{lk} = P^a_p g_{al}, \]

\[ K^a_p = K^a_{kp} g_{al}, \quad E_{kpq} = E^a_{kp} g_{ak}, \quad T_{lkp} = T^a_{kp} g_{al}, \]

\[ M_{pqr} = T_{pqr} + T_{qrp} + T_{rqp}. \]

Moreover, for any $(0, 2)$ tensor $T$ we put

\[ \overline{T}_{ab} = T_{ab} + T_{ba}, \quad \hat{T}_{ab} = T_{ab} - T_{ba}. \]

Lemmas 3.2-3.9 were proved in ([9], see also [10]). Hereafter, and unless otherwise specified, all the coefficients $a_j, b_j, a'_j, b'_j, A, A', B, B', ...$ are considered to be constants, equal to the values at 0 of the corresponding functions.
Lemma 3.2. Under hypothesis (3.1) at a point \((x,0) \in TM\) we have:
\[
a_1T_{ikp} + a_2E_{ikp} = a'_1(Y_ig_{kp} - Y_kg_{ip} - Y_pg_{ki}) - b_1Y_ig_{kp},
\]
(3.4) \(AE_{ikp} + a_2T_{ikp} + a'_2(g_{ki}Y_p + g_{pi}Y_k) + \frac{1}{2} b_2(2g_{kp}Y_i + g_{ip}Y_k + g_{ki}Y_p) = 0.\)

If \(a \neq 0\), then
\[
aE_{ikm} = (a_2b_1 - a_1b_2 - 2a_2a'_1)g_{km}Y_i - \frac{1}{2}(a_1b_2 - 2a_2a'_1 + 2a_1a'_2)(g_{im}Y_k + g_{ik}Y_m),
\]
(3.5) \(aE_{ikm} = \frac{1}{2}(a_1b_2 - 2a_2a'_1 + 2a_1a'_2)(g_{im}Y_k + g_{ik}Y_m),\)
(3.6) \(aT_{ikm} = (a_1b_2 - 2ab_1)g_{km}Y_i + \frac{1}{2}(a_2b_2 - 2a_1b_1)(g_{im}Y_k + g_{ik}Y_m),\)
(3.7) \(aM_{ikm} = [2a_2(b_2 + a'_2) - A(b_1 + a'_1)](g_{km}Y_i + g_{ik}Y_m + g_{mi}Y_k).\)

Moreover,
\[
a_2[\nabla_k(\nabla_iX_p + \nabla_pX_i) + \nabla_i(\nabla_kX_p + \nabla_pX_k) - \nabla_p(\nabla_iX_k + \nabla_kX_i) + a_1(\nabla_k\nabla_iY_p + \nabla_i\nabla_kY_p)] = 2A'g_{kl}Y_p + B(Y_ig_{kp} + Y_ig_{kp}),
\]
(3.8) \(a(\nabla_kK_{kp} + \nabla_iK_{kp}) + (a_2b_2 + 2a_1A' - 2a_2a'_2)Y_pg_{kl} + \)
\[
\frac{1}{2}(-a_2b_2 + 2a_1B + 2a_2a'_2)(Y_ig_{kp} + Y_i(g_{kp}) = 0.
\]

Lemma 3.3. Under hypothesis (3.1) we have
\[
2a\nabla_iK_{km} = a_1^2Y^rb_{rmkl} - a_1Bg_{km}Y_i + \)
\[
(-a_1B + a_2b_2 - 2a_2a'_2)g_{im}Y_k + (-a_2b_2 - 2a_1A' + 2a_2a'_2)g_{kl}Y_m,
\]
(3.10) \(2a(\nabla_iS_{km} - X^rR_{rkm}) + a_1a_2Y^rR_{rkm} - a_2Bg_{km}Y_i + [a_2B + A(b_2 - 2a'_2)]g_{im}Y_k + [-a_2A' - A(b_2 - 2a'_2)]g_{kl}Y_m = 0\)
at the point.

Lemma 3.4. Under hypothesis (3.1) suppose \(\dim M > 2\). Then on \(M \times \{0\}\)
\[
T_{kl} = T_{ik} = 2(b_1 - a'_1)\overline{S}_{kl} + b_2\overline{K}_{kl} = 0,
\]
\[
a_2F_{labk} + a_1W_{labk} + \frac{1}{2} b_2 \left( \overline{K}_{lak}g_{ab} + \overline{K}_{lab}g_{ak} + \overline{K}_{alb}g_{ak} + \overline{K}_{akb}g_{ab} \right) +
\]
\[
b_1g_{kl}S_{ak} + a'_1(g_{kl}S_{ab} + g_{al}S_{bk}) = 0.
\]

Lemma 3.5. Under hypothesis (3.1) suppose \(\dim M > 1\). Then
\[
(n - 1)\beta Y_i = 0
\]
on \(M \times \{0\}\) holds, where
\[
\beta = 2A(b_1^2 - a_2^2 - a_1b_1) + (a_1b_2 - 2a_2b_1)(3b_2 + 2a'_2) + 2a_2[2a'_1(b_2 + a'_2) + a_2b'_1].
\]

Corollary 3.1. For the Cheeger-Gromoll metric \(g^{CG}\) on \(TM\), the vector field \(Y\) vanishes everywhere on \(M\).
Lemma 3.6. Under hypothesis (3.1) the identities

\[ 3AF_{kmn} + 3a_2 W_{kmn} + B \left( g_{kt} \bar{F}_{mn} + g_{lm} \bar{F}_{kn} + g_{ln} \bar{F}_{km} \right) + (b_1 - a'_1) (Y_{nt} g_{km} + Y_{ml} g_{kn} + Y_{lk} g_{mn}) + 2(b_2 + a'_2) \left( g_{kl} \bar{S}_{mn} + g_{lm} \bar{S}_{kn} + g_{ln} \bar{S}_{km} \right) + 2b_2 \left( g_{kn} (X_{n,t} + S_{ln}) + g_{kn} (X_{m,l} + S_{lm}) + g_{mn} (X_{k,l} + S_{lk}) \right) = 0 \]

and

\[ (3.12) \quad B \left[ g_{kt} (K_{mn} - 2K_{nm}) + g_{lm} (K_{kn} + K_{nk}) + q_{kn} (K_{mk} - 2K_{km}) \right] + 2(b_1 - a'_1) (2Y_{nt} g_{km} - Y_{nl} g_{kn} - Y_{lk} g_{mn}) + 3a_1 (K_{pqr} R_{kmn}^{pqr} + K_{pqr} R_{mkn}^{pqr}) + b_2 \left[ 2g_{kn} (X_{m,l} + S_{lm}) - g_{km} (X_{n,t} + S_{ln}) - g_{mn} (X_{k,l} + S_{lk}) \right] + (b_2 - 2a'_2) \left( 2g_{lm} \bar{S}_{kn} - g_{ln} \bar{S}_{km} - g_{kl} \bar{S}_{mn} \right). \]

are satisfied at a point \((x, 0) \in TM\).

Lemma 3.7. Under hypothesis (3.1) relation

\[ (3.13) \quad 3a_2 \left[ E_{bc}^p (R_{pkl} + R_{lkb}) + E_{ac}^p (R_{pkl} + R_{lkb}) + E_{ab}^p (R_{pkl} + R_{lkb}) \right] + 6A' g_{kl} (T_{abc} + T_{a} + T_{abc}) + g_{bc} K_{kal} + g_{ca} K_{kal} + g_{ab} K_{kal} + g_{kl} L_{abk} + g_{bl} L_{bck} + g_{bk} L_{cak} + g_{ak} L_{abl} + g_{bk} L_{cal} = 0 \]

holds on \(M \times \{0\}\), where

\[ (3.14) \quad K_{kal} = K_{lak} = -2b_2 (S_{ka,t} + S_{la,k} + X_{a,kl} + X_{a,lk}) - (b_1 - a'_1) (Y_{a,kl} + Y_{a,lk}), \]

and

\[ (3.15) \quad L_{abk} = L_{bak} = 2B \bar{K}_{ab,k} + 3BT_{abk} + (b_2 - 2a'_2) \bar{S}_{ab,k} + 3B' (g_{ka} Y_{k} + g_{kb} Y_{b}). \]

Lemma 3.8. Under hypothesis (3.1) suppose \(\dim M > 2\). Then the relation

\[ a_1 \left[ 2E_{ab}^p R_{pkl}^c - E_{bc}^p R_{pkl}^a + E_{ac}^p R_{pkl}^b - E_{ab}^p R_{pkl}^c + E_{ac}^p R_{pkl}^b \right] + B \left[ (E_{cbb} + E_{bcb}) g_{at} + (E_{cak} + E_{bkc}) g_{bt} + \right. \right. \]

\[ (E_{abc} + E_{cab}) g_{ct} - (E_{abc} + E_{bac}) g_{kt} \right] + (b_1 - a'_1) \left[ \nabla_i \bar{S}_{bc} g_{ak} - \nabla_i \bar{S}_{bk} g_{ac} \right] + \]

\[ b_2 \left[ \nabla_i \bar{K}_c g_{ab} + g_{ak} \left( \frac{3}{2} \nabla_i \bar{K}_{bc} + \frac{1}{2} \nabla_i \bar{K}_{cb} \right) - g_{ac} \left( \frac{3}{2} \nabla_i \bar{K}_{bk} + \frac{1}{2} \nabla_i \bar{K}_{kb} \right) \right] + \]

\[ b_2 \left[ \nabla_i \bar{K}_{ac} g_{bk} - \nabla_i \bar{K}_{ak} g_{bc} \right] + \]

\[ (b_2 - 2a'_2) \left[ M_{abk} g_{ct} - M_{abc} g_{kt} \right] + b_2 \left[ g_{ka} T_{a} + g_{kb} T_{b} + g_{ak} T_{abk} + g_{bk} T_{bck} - g_{ac} T_{lck} \right] + \]

\[ 2b_2' \left[ (g_{ka} g_{ct} - g_{kc} g_{at}) Y_a + (g_{ak} g_{ct} - g_{ac} g_{kt}) Y_b \right] + \]

\[ \left. (g_{al} g_{ak} + g_{ak} g_{al}) Y_{c} - (g_{al} g_{ak} + g_{ac} g_{kt}) Y_k \right] = 0 \]

holds on \(M \times \{0\}\).
Lemma 3.9. Under hypothesis (3.1) relations

\begin{align*}
A_{km} &= (3a_1B - a_2b_2)\nabla_kX_m + (-2a_2b_1 + \frac{3}{2}a_1b_2 + 2a_2a_1' - 3a_1a_2')\nabla_kY_m + \\
a_2B(K_{km} - 2K_{mk}) + (3a_1B - 2a_2b_2 + 2a_2a_1')S_{km} + \\
(-a_2b_2 + 2a_2a_1'S_{mk} = 0, \\
F_{kl} + B_{kl} &= 2a_2b_2(L_Xg)_{kl} + (4a_2b_1 - 3a_1b_2 - 4a_2a_1')(L_Yg)_{kl} + \\
&2(3a_2b_2 + 3a_1A' - 4a_2a_1')S_{kl} + 2a_2B\mathcal{K}_{kl} = 0.
\end{align*}

hold at a point \((x,0) \in TM\),

\begin{align*}
F_{mn} &= 2a_2B\mathcal{K}_{mn} + 2(2a_2b_2 + 3a_1A' - 4a_2a_1')S_{mn},
B_{kl} &= 2a_2b_2(L_Xg)_{kl} + (4a_2b_1 - 3a_1b_2 - 4a_2a_1')(L_Yg)_{kl} + 2a_2b_2S_{kl}.
\end{align*}

4. Classification

4.1. Case 1. In this section we study relations between \(Y\) component of the Killing vector field on \(TM\) and the base manifold \(M\) (Theorems 4.1, 4.2). Various conditions for \(Y\) to be non-zero and relations between \(X, Y, P, K\) are proved. Moreover, Theorem 4.3 establishes isomorphism between algebras of Killing vector fields on \(M\) and \(TM\) for a large subclass of non-degenerate \(g\)-natural metrics.

Lemma 4.1. Under hypothesis (3.1) suppose \(\dim M > 2\) and \(2(b_1 - a_1')a_2 - a_2b_2 \neq 0\) at a point \((x,0) \in TM\). Then

\begin{align*}
(B + A')Y_k &= 0, \\
2a\nabla_lK_{km} &= [2a_1A' + a_2(b_2 - 2a_1')](g_{lm}Y_k - g_{lk}Y_m), \\
2a\nabla_lP_{km} &= [-2a_2A' + A(b_2 - 2a_1')](g_{lm}Y_k - g_{lk}Y_m), \\
a_1\nabla_m\nabla_lY_k &= A'(g_{ml}Y_k - g_{mk}Y_l), \\
a_1Y^rR_{rklm} &= A'(g_{km}Y_l - g_{kl}Y_m)
\end{align*}

hold at the point.

Proof. First suppose \(a_1 \neq 0\). Symmetrizing (3.10) in \((k,m)\), making use of the skew-symmetry of \(K\), then alternating in \((k,l)\) and applying the first Bianchi identity, we get

\begin{align*}
3a_1Y^rR_{rklm} + (B - 2A')(g_{lm}Y_k - g_{km}Y_l) &= 0.
\end{align*}

Applying the last identity to (3.10) we find

\begin{align*}
6a\nabla_lK_{km} + 2a_1(B + A')g_{km}Y_l + 3[2a_1A' + a_2(b_2 - 2a_1')]g_{lk}Y_m + \\
[2a_1(2B - A') - 3a_2(b_2 - 2a_1')]g_{lm}Y_k &= 0,
\end{align*}

whence, symmetrizing in \((k,m)\), we obtain (4.1) and, consequently, (4.2).

Suppose now \(a_1 = 0\). Substituting in (3.10) we easily state that (4.2) remains true. On the other hand, substituting \(a_1 = 0\) into (3.11) and symmetrizing in \((k,m)\) we get

\begin{align*}
2a_2Bg_{km}Y_l + a_2(B + 2A')(g_{lm}Y_k + g_{lk}Y_m) &= 0,
\end{align*}
whence, by contractions with $g^{km}$ and $g^{lm}$, we obtain

\begin{equation}
BY_1 = 0 \quad \text{and} \quad A^i Y_1 = 0
\end{equation}

respectively since $a_2 \neq 0$ must hold. Thus (4.1) holds good.

Since $X$ is a Killing vector field, (3.11), (2.2), (4.1) and (4.6) in the case $a_1 \neq 0$ and (3.11) and (4.7) as well in the case $a_1 = 0$ yield (4.3).

Differentiating covariantly (equation II1, [9]) and using just obtained identities, we get (4.4). Finally, alternating (4.4) in $(l, m)$, by the use of the Ricci identity (2.1), we obtain (4.5). This completes the proof. \hfill $\Box$

From (4.5) and Theorem 6.1 by Grycak we infer

\textbf{Theorem 4.1.} Under hypothesis (3.1) suppose $\dim M > 2$ and $2(b_1 - a'_1) a_2 - a_1 b_2 \neq 0$ on the set $M \times \{0\} \subset TM$. If the vector field $\frac{\partial}{\partial X} Y^a \partial_a$ does not vanish on a dense subset of $M$ and $M$ is semisymmetric, i.e. $R \cdot \hat{R} = 0$, (resp. the Ricci tensor $S$ is semisymmetric, i.e. $R \cdot S = 0$), then $M$ is a space of constant curvature, (resp. $M$ is an Einstein manifold).

\textbf{Theorem 4.2.} Under hypothesis (3.1) suppose $\dim M > 2$ and $2(b_1 - a'_1) a_2 - a_1 b_2 \neq 0$ at a point $(x, 0) \in TM$. Then the $Y$ component of the Killing vector field on $TM$ satisfies

\begin{equation}
S_1 Y \left[ a_1 R + \frac{B}{2} g \wedge g \right] = 0
\end{equation}
on $M$.

\textit{Proof.} Suppose $a_1 \neq 0$. By (1.1) and (1.2) we have $S_{ab} = 0$. Applying this and (1.2), (4.1), (4.2) and (4.7) to Lemma 3.8, after long computations we obtain

\begin{equation}
S_1 \left[ 3(R_{blck} Y_a + R_{alc} K_b) + (R_{blck} + R_{alc}) Y_c - (R_{blac} + R_{alc}) Y_k \right] +
S_2 g_{ab} (g_{dl} Y_c - g_{cl} Y_k) + S_3 [(g_{al} g_{bk} + g_{ak} g_{bl}) Y_c - (g_{al} g_{bc} + g_{ac} g_{bl}) Y_k] +
S_4 [(g_{bk} g_{cl} - g_{bc} g_{kl}) Y_a + (g_{ak} g_{cl} - g_{ac} g_{kl}) Y_b] = 0,
\end{equation}

where

\begin{align*}
S_1 &= a_1 [2a_2 a'_1 - a_1 (b_2 + 2a'_2)], \\
S_2 &= -2 [b_2 (-Ab_1 + 3a_2 b_2 + 5a_1 A' - Aa'_1 - 4a_2 a'_2) + 2b_1 (Aa'_2 - a_2 A')] +
2(a_1 A' + Aa'_1 - 2a_2 a'_2) a'_2] =
-2 [b_2 (-Ab_1 + 3(a_2 b_2 + a_1 A' - Aa'_1) + 2a') + b_1 (Aa'_2 - a_2 A') + 2a' a'_2],
\end{align*}

\begin{align*}
S_3 &= -3a_1 b_2 A' - 2Ab_2 a'_1 + 2a_2 A' a'_1 + 4a_2 a'_2 b_2 - 2a_1 A' a'_2 + 4ab'_2 = 2A' (a_2 a'_1 - a_1 a'_2) - b_2 (2a' + a_1 A') + 4ab'_2,
\end{align*}

\begin{align*}
S_4 &= b_2 (-2Ab_1 + 6a_2 b_2 + 7a_1 A' - 4Aa'_1 - 4a_2 a'_2) - 4a_2 b_1 A' + 2a_2 A' a'_1 +
a'_2 (4Ab_1 + 2a_1 A' + 4Aa'_1 - 8a_2 a'_2) + 4ab'_2
\end{align*}

and

\begin{align*}
S_2 - S_3 + S_4 &= 0
\end{align*}

identically.
Symmetrizing (4.9) in \((a, b, l)\) we get
\[
(S_2 + 2S_3) \left( (gkl g_{b} + g_{a} g_{k} b + g_{a b} g_{k l}) Y_{c} - 2 (g_{a} g_{b} + g_{a c} g_{t} + g_{a b} g_{a l}) Y_{k} \right) = 0,
\]
whence, by contraction with \(g^{ab}\), we find \((n - 1)(n + 2)(S_2 + 2S_3) Y_c = 0\). Therefore, symmetrizing (4.9) in \((\alpha, \beta, \gamma)\) and using the last result, we obtain
\[
Y_{a} T_{bc kl} + Y_{b} T_{ca kl} + Y_{c} T_{ab kl} = 0,
\]
where
\[
T_{bc kl} = T_{cb kl} = T_{k b c l} = 2 S_1 (R_{bkcl} + R_{bckl}) - (S_3 + S_4) \left( g_{bc} g_{kl} - \frac{1}{2} (g_{a} g_{b} + g_{a c} g_{t} + g_{a b} g_{a l}) \right).
\]
Hence, by the use of the Walker’s Lemma 6.1, we get
\[
(4.10) \quad Y_{a} T_{bc kl} = 0.
\]
Alternating (4.10) in \((l, c)\) and applying the Bianchi identity we obtain
\[
Y_{a} \left[ 4 S_1 R_{bkcl} + (S_3 + S_4) (g_{a} g_{b} + g_{a c} g_{t} + g_{a b} g_{a l}) \right] = 0.
\]
Transvecting the last equation with \(Y^b\), by the use of (4.7), we easily get
\[
[4 B S_1 + a_1 (S_3 + S_4)] Y_{a} = 0,
\]
whence (4.8) results.

On the other hand, from the proof of Lemma 4.1 it follows that \(a_1(0) = 0\) implies \(B(0) Y_{a} = 0\). Thus, by continuity, (4.8) holds good on \(M\).

**Corollary 4.1.** Under assumptions of the above theorem we have on \(M\) :

\[
(S_2 + 2S_3) Y = 0 \text{ if } a_1 \neq 0,
\]
\[
[4 B S_1 + a_1 (S_3 + S_4)] Y = 0.
\]

Notice that multiplying the first equation by \(a_1\) and adding to the second one we obtain
\[
a_1 (b_2 a' - 2 a b_1) Y = 0.
\]

**Lemma 4.2.** Under hypothesis (3.1) suppose \(\dim M > 2\) and \(2 (b_1 - a'_1) a_2 - a_1 b_2 \neq 0\) at a point \((x, 0) \in T M\).

If \(a_1 a_2 \neq 0\), then
\[
(4.11) \quad A_{km} = \left[ 2 a_2 (b_1 - a'_1) - 3 a_1 (b_2 - 2 a') \right] Y_{k, m} + (3 a_1 B - a_2 b_2) P_{km} + 3 a_2 B K_{km} = 0.
\]

If \(a_2 = 0\) and \(a_1 b_2 \neq 0\) then
\[
(4.12) \quad \frac{1}{3} A_{km} = - \frac{1}{2} a_1 (b_2 - 2 a') Y_{k, m} + a_1 B P_{km} = 0.
\]

If \(a_1 = 0\) and \((b_1 - a_1') a_2 \neq 0\) then
\[
(4.13) \quad (n + 1) B K_{kn} - b_2 P_{kn} + 2 (b_1 - a'_1) Y_{k, n} = 0,
\]
\[
(4.14) \quad 3 B K_{ln} - (n - 1) b_2 P_{ln} + 2 (n - 1) (b_1 - a'_1) Y_{l, n} = 0.
\]
Proof. If $a_1a_2 \neq 0$, we apply (1.1) and (1.2) to (3.16) to obtain (4.11).

If $a_2 = 0$ but $a_1 \neq 0$, then also there must be $b_2 \neq 0$. Substituting $a_2 = 0$ into (3.16) and applying (1.1) and (1.2) we get (4.12).

Finally, the last two identities one obtains substituting $a_1 = 0$ into (3.12), contracting with $g^{km}$ and $g^{lm}$ and making use of (1.1) and (1.2).

Taking into account (4.13) and (4.14) together with the equation (equation $II_1$, [9]) which, in virtue of (1.1), writes

$$AK_{km} + a_2 P_{km} - a_1 Y_{k,m} = 0$$

we find that $B \neq 0$ implies $P = K = \nabla Y = 0$ on $M$. We conclude with the following

**Theorem 4.3.** Let $TM, \dim TM > 4$, be endowed with a non-degenerate $g$- natural metric $G$, such that $A_1 = 0$, $(b_1 - a_1')a_2 \neq 0$ and $B \neq 0$ on $M \times \{0\} \subset TM$. Let $V$ be an open subset of $TM$ such that $M \times \{0\} \subset V$. If $V$ admits a Killing vector field, then it is a complete lift of a Killing vector field on $M$. Consequently, Lie algebras of Killing vector fields on $M$ and $V \subset TM$ are isomorphic.

Besides, for $B = 0$, we have

**Theorem 4.4.** Let $TM, \dim TM > 4$, be endowed with a non-degenerate $g$- natural metric $G$, such that $a_1 = 0$, $(b_1 - a_1')a_2 \neq 0$ and $B = 0$ on $M \times \{0\} \subset TM$. Then

$$a_2 P + AK = 0,$$

$$b_2 P - 2(b_1 - a_1')\nabla Y = 0$$

hold on $M \times \{0\} \subset TM$.

Hence, for $B = 0$, $A \neq 0$ and $b_2 \neq 0$, a theorem similar to the former one can be deduced.

The next theorem gives further restrictions on the vector $Y$ to be non-zero.

**Theorem 4.5.** Under hypothesis (3.1) suppose $\dim M > 2$ and $2(b_1 - a_1')a_2 - a_1b_2 \neq 0$ at a point $(x,0) \in TM$. If $a_1 \neq 0$, then the $Y$ component of the Killing vector field on $TM$ satisfies

$$Q_2 Y = \{a_1b_2[A(b_2 - 2a_2') - 2a_2B] - 4aB(b_1 - a_1')\} Y = 0,$$

$$B' Y = 0,$$

$$B[a_1a_2(b_2 + 2a_2') - 2Aa_1a_1' + aa_1'] Y = 0.$$

Proof. We apply Lemma 3.7. By the use of (1.1), (1.2), (4.1) - (4.4) and (3.6) the components of the tensors $K$ and $L$ defined by (3.14) and (3.15) can be written as

$$K_{kat} = \frac{[aB(b_1 - a_1') + 2a_1a_2Bb_2 - Aa_1b_2(b_2 - 2a_2')]}{aa_1}(2g_{ka}Y_a - g_{ka}Y_t - g_{ta}Y_k),$$

$$L_{abd} = 3BT_{lab} + 3B'(g_{ab}Y_a + g_{ad}Y_b) =$$

$$= \frac{3B[A(b_1 - a_1') - a_2b_2]}{a} g_{ab}Y_t +$$

$$+ \frac{3[B(a_2b_2 - 2Aa_1' + 2a_2a_2') + 2aB']}{2a} (g_{ad}Y_b + g_{bd}Y_a).$$
Substituting into (3.13) and applying (3.5), (3.7) and (4.7) we get

\[ \text{(4.15)} \quad Q_1 [(R_{bcl} + R_{bock}) Y_a + (R_{ckal} + R_{akbl}) Y_b + (R_{akbl} + R_{abck}) Y_c] + \]
\[ Q_2 [(g_{ab} g_{hc} + g_{al} g_{ca} + g_{ld} g_{ab}) Y_a + (g_{ak} g_{hb} + g_{bk} g_{ca} + g_{ck} g_{ab}) Y_l + \]
\[ Q_3 g_{kl} (g_{bc} Y_a + g_{ca} Y_b + g_{ab} Y_c) + \]
\[ Q_4 [(g_{kl} g_{hc} + g_{lk} g_{lc}) Y_a + (g_{al} g_{ca} + g_{ck} g_{la}) Y_b + (g_{al} g_{bk} + g_{ck} g_{lb}) Y_c] = 0, \]

where
\[ Q_1 = -3a_2 (a_1 b_2 - 2a_2 a_1' + 2a_1 a_2'), \]
\[ Q_2 = [a_1 b_2 (A(b_1 - a_1') - 2B a_2) - 4a B (b_1 - a_1')] \]
\[ Q_3 = 24 \frac{A B (b_1 - a_1') - a_1 [A(b_2 - 2a_1') + B(a_2 b_2 - 6a_1' + 6a_2 a_1')]}{a a_1}, \]
\[ Q_4 = 3B (a_2 b_2 - 2a_1' + 2a_2 a_1') + 2a B'. \]

Contracting (4.15) with \( g^{ab} \), by the use of (4.7), we get

\[ \text{(4.16)} \quad g_{kl} \left( -\frac{4B Q_1}{a_1} + (n + 2)Q_3 + 2Q_4 \right) Y_c - 2Q_4 R_{kl} Y_c + \]
\[ \left( \frac{2B Q_1}{a_1} + (n + 2)Q_2 + 2Q_4 \right) \left( g_{il} Y_k + g_{kc} Y_l \right) = 0. \]

Symmetrizing in \((c, k, l)\) we obtain
\[ T_{kl} Y_c + T_{lc} Y_k + T_{ck} Y_l = 0, \]

where
\[ \text{(4.17)} \quad T_{kl} = T_{lk} = g_{kl} [(n + 2) (2Q_2 + Q_3) + 6Q_4] - 2Q_4 R_{kl}. \]

Then the Walker lemma yields \( T_{kl} = 0 \) or \( Y_c = 0 \). Subtracting (4.17) from (4.16) and contracting with \( g^{kl} \) we get

\[ \text{(4.18)} \quad [a_1 ((n + 2)Q_2 + 2Q_4) + 2B Q_1] Y_c = 0. \]

In the same way, by contraction of (4.15) with \( g^{kl} \), we find

\[ \text{(4.19)} \quad \{g_{bk} [(n + 5)Q_2 + 3Q_3 + 2(n + 2)Q_4] + 2Q_1 R_{bk}\} Y_c = 0 \]

and
\[ \text{(4.20)} \quad [a_1 ((n + 3)Q_2 + Q_3) - 2B Q_1] Y_c = 0. \]

At last, by contraction of (4.15) with \( g^{kl} \), we obtain
\[ \text{(4.21)} \quad [g_{hc} (2Q_2 + nQ_3 + 2Q_4) - 2Q_1 R_{hc}] Y_a = 0. \]

Eliminating the Ricci tensor between (4.17), (4.21) and (4.19) we find
\[ [3(n + 3)Q_2 + (n + 5)(Q_3 + 2Q_4)] Y_c = 0, \]
\[ [(n + 1)Q_2 + 2Q_3 + 2Q_4] Y_c = 0. \]

The system consisting of (4.18), (4.20) and the above two equations is indetermined and equivalent to \( Y = 0 \) or \( Q_2 = 0 \) and \( 2B Q_1 + a_1 Q_3 = 0 \) and \( Q_3 + 2Q_4 = 0 \). Hence \( 2Q_2 + Q_3 + 2Q_4 \) yields the second identity, while \( a_1 (Q_3 + 2Q_4) - (2B Q_1 + a_1 Q_3) \) gives the third one. \( \square \)
Proof. It follows from the results of subsection 5.3.3.

Let $\pi$ be an infinitesimal isometry.

First consider the case

4.2. Case 2. The next theorem partially improves the result of Tanno ([17]) concerned with Killing vector field on $(TM, g^C)$, where the complete lift $g^C$ of $g$ is a $g$-natural metric with $a_2 = 1$, all others being zero. (In Tanno’s paper the Killing vector on $(TM, g^C)$ is of the form $\iota C[X] + X^C + Y^c + (\alpha^t P^t_i)\partial_i$, where $Y$ and $P$ satisfy some additional conditions). Furthermore, we prove in the section some sufficient conditions for $X$ and $Y$ to be either infinitesimal affine transformation or infinitesimal isometry.

Theorem 4.6. Let $X$ be an infinitesimal affine vector field on some open $U \subset M$. If $a_2 = \text{const} \neq 0$, $b_3 = \text{const}$, all others equal 0 on $\pi^{-1}(U) \subset TM$, then $\iota C[X] + X^C$ is a Killing vector field on $\pi^{-1}(U)$.

Proof. It follows from the results of subsection 5.3.3. \qed

Lemma 4.3. Under hypothesis (3.1) suppose $\dim M > 2$ and $2(b_1-a_1')a_2-a_1b_2 = 0$ at a point $(x,0) \in TM$. Moreover, let either $a_1a_2b_2 \neq 0$ or $a_2 \neq 0$, $b_2 = 0$, $b_1 - a_1' = 0$. Then

$$(a_1B - 2a_2b_2 - 3a_1A' + 4a_2a_2') \left[ (L_X g) - \frac{1}{n} Tr(L_X g) g \right] = 0,$$

$$a_2(b_1 - a_1') \left[ (L_Y g) - \frac{1}{n} Tr(L_Y g) g \right] = 0,$$

$$a_1 \left[ a_2' (L_Y g) + A' (L_X g) \right] = 0,$$

$$[a_1 (B - 3A') + A(b_1 - a_1') - 2a_2(b_2 - 2a_2')] (L_X g) = 0.$$

Proof. First consider the case $a_1a_2b_2 \neq 0$. By the use of (1.3) - (1.5) and the equality $a_1b_2 = 2a_2(b_1 - a_1')$ Lemma 3.9 yields

$$
F = 2(a_1B - 2a_2b_2 - 3a_1A' + 4a_2a_2') (L_X g),
$$

$$
B = -2a_2(b_1 - a_1') (L_Y g),
$$

whence, by ([9], Lemma 19 or [10], Lemma 54), the first two equalities result. Moreover, by Lemma 3.9 we have

$$(4.22) \quad F + B = -2a_2(b_1 - a_1') (L_Y g) + 2(a_1B - 2a_2b_2 - 3a_1A' + 4a_2a_2') (L_X g) = 0,$$

and

$$A_{km} = 3a_2BK_{km} + (3a_1B - a_2b_2) P_{km} + (a_1B - 2a_2a_2') (L_X g)_{km} +
[a_2 (b_1 - a_1') - 3a_1a_2'] \nabla_k Y_m = 0.$$

Symmetrizing in $(k, m)$ and transforming the obtained equation in the same manner as before we find

$$(4.23) \quad [a_2 (b_1 - a_1') - 3a_1a_2'] (L_Y g) - (a_1B - 2a_3b_2 + 4a_2a_2') (L_X g) = 0.$$

Now from (4.22) and (4.23) we easily deduce the third equality. Finally, the last one is obtained by applying (1.4) to (4.22).

The proof of the second case can be obtained in the same way. The statements differ only in that $b_2 = 0$. \qed
Corollary 4.2. If \( a_1 (a_2 A' - a'_2 A) \neq 0 \), then \( L_X g = 0 \).

4.3. Case 3. The main result of the section establishes isomorphism between algebras of Killing vector fields on \( M \) and \( TM \) for a large subclass of \( g \)-metrics (Theorem 4.7). Furthermore, conditions for \( \bar{Y} \) to be non-zero are proved.

Lemma 4.4. Under hypothesis (3.1) suppose that \( \dim M > 2 \) and the following conditions on \( a_j, b_j \) at a point \( (x, 0) \in M \) are satisfied: \( a_1 = 0, b_1 = a'_1, a_2 \neq 0, b_2 \neq 0 \). Then the relations

\[
\begin{align*}
(b_2 - 2a'_2) L_X g &= 0, \\
(b_2 - 2a'_2) Tr(\nabla X) &= 0, \\
(b_2 - 2a'_2) TrP &= 0,
\end{align*}
\]

(4.25) \( BK = 0 \), \( L_X g + P = 0 \)

hold. Moreover \( P \) is symmetric. Finally \( a_3 K = 0 \).

Proof. Substituting \( a_1 = 0 \) and \( a'_1 = b_1 \) into (3.12), then applying (1.8) and (1.6) we find

\[
\begin{align*}
&b_2 [2g_{kn} ((L_X g)_{lm} + P_{lm}) + g_{mn} ((L_X g)_{kl} + P_{kl}) - g_{km} ((L_X g)_{lm} + P_{lm})] + \\
g_{ln} - 3BK_{km} + (b_2 - 2a'_2) (L_X g)_{km} + \\
g_{kl} [3BK_{mn} + (b_2 - 2a'_2) (L_X g)_{mn}] - 2(b_2 - 2a'_2) g_{im} (L_X g)_{km} = 0.
\end{align*}
\]

(4.26)

From (1.6) it follows that \( P^a_n + 2X_n^a = 0 \). Thus contracting (4.26) with \( g^{lm} \) and then with \( g^{kn} \) we get (4.24) in turn. Consequently, contracting (4.26) with \( g^{kn} \), by the use of (1.6), (1.8) and (4.24), we obtain

\[
-3BK_{lm} + (n - 1)b_2 [P_{lm} + (L_X g)_{lm}] = 0.
\]

In a similar way, contracting (4.24) with \( g^{kl} \), we find

\[
-(n + 1)BK_{mn} + b_2 [P_{mn} + (L_X g)_{mn}] = 0.
\]

The last two equations yield (4.25). The final statement is a consequence of (4.25), (equation \( II_1 \), [9] ) and \( a_1 = 0 \). \( \square \)

Lemma 4.5. Under assumptions of Lemma 4.4 relations

\[
\begin{align*}
[(b_2 - 2a'_2) (2Ab_1 - 3a_2 b_2 - 2a_2 a'_2) - 2a_2 Bb_1] Y &= 0, \\
[a_2 Bb_1 + Ab_1 b_2 - 2a_2 (b_2 a'_2 - a_2 b'_2)] Y &= 0, \\
(b_1 b_2 - a_2 b'_2) Y &= 0
\end{align*}
\]

hold on \( M \times \{0\} \).

Proof. We apply Lemma 3.8. Substituting \( a'_1 = b_1, a_1 = 0 \), contracting with \( g^{ab} g^{cl} \) and applying (1.8) we get

\[
-2b_2(n + 2)K^r_{kr} + 2BE^r_{lr} - 2BE^r_{kr} + (n - 1)(b_2 - 2a'_2) M^r_{kr} = 0,
\]

whence, by the use of Lemma 3.2 we obtain the first equality. Similarly, contracting with \( g^{al} g^{bc} \) we find

\[
-b_2(n + 2)K^r_{kr} + B(n + 2) E^r_{kr} - B(n + 2) E^r_{kr} - b_2 n T^r_{kr} + b_2 T^r_{kr} - 2(n + 2)(n - 1) Y_k = 0,
\]

whence the second equation results. Finally, the third one follows from Lemma 3.5. \( \square \)
Lemma 4.6. Under assumptions of Lemma 4.4 suppose \( L_X g = 0 \). Then
\[ AY = BY = A'Y = 0 \]
at each point \((x,0) \in TM\).

Proof. By (1.7), \( Y \) is a Killing vector field on \( M \). Moreover, (3.8) reduces to
\[ 2A'g_k Y_p + B(Y_k g_p + Y_l g_{kp}) = 0, \]
whence we easily deduce \( BY = A'Y = 0 \). Since an infinitesimal isometry is also an infinitesimal affine transformation, from (3.11), by the use of (2.2) and the above properties, we obtain \( AY = 0 \). \( \square \)

Lemma 4.7. Under assumptions of Lemma 4.4 suppose
\begin{equation}
L = 0.
\end{equation}

Then \( \nabla L = 0 \) if and only if \( BY = A'Y = 0 \).

Proof. Substituting into (3.8), symmetrizing in \((k,p)\) and applying (4.27) we get
\[ a_2 \nabla Y_p + B(2g_k Y_l + g_p Y_k + g_{kp}) + 2A'(g_k Y_p + g_l Y_p) = 0, \]
whence the thesis results. \( \square \)

A complete lift of a Killing vector field on \( M \) to \((TM,G)\) is always a Killing vector field ([9], [10]). Thus we have proved

Theorem 4.7. Let on \( TM \), \( \dim TM > 4 \), a \( g \)-natural metric \( G \)
\begin{align*}
G_{(x,u)}(X^h, Y^h) &= A(r^2)g_x(X,Y) + B(r^2)g_x(X,u)g_x(Y,u), \\
G_{(x,u)}(X^h, Y^v) &= a_2(r^2)g_x(X,Y) + b_2(r^2)g_x(X,u)g_x(Y,u), \\
G_{(x,u)}(X^v, Y^h) &= a_2(r^2)g_x(X,Y) + b_2(r^2)g_x(X,u)g_x(Y,u), \\
G_{(x,u)}(X^v, Y^v) &= 0
\end{align*}
be given, where \( a_2 b_2 \neq 0 \) everywhere on \( TM \) while \( b_2 - a_2' \) and either \( A \) or \( B \) do not vanish on a dense subset of \( TM \). If \( Z \) is a Killing vector field on \( TM \), then there exists an open subset \( U \) containing \( M \) such that \( Z \) restricted to \( U \) is a complete lift of a Killing vector field \( X \) on \( M \), i.e.
\[ Z|_U = X^C. \]

4.4. Case 4. The class under consideration contains the Sasaki metric \( g^S \) and the Cheeger-Gromoll one \( g^{CG} \). In ([18]) Tanno proved the following

Theorem 4.8. Let \((M,g)\) be a Riemannian manifold. Let \( X \) be a Killing vector field on \( M \), \( P \) be a \((1,1)\) tensor field on \( M \) that is skew-symmetric and parallel and \( Y \) be a vector field on \( M \) that satisfies \( \nabla_k \nabla_l Y_p + \nabla_l \nabla_k Y_p = 0 \) and (4.31). Then the vector field \( Z \) on \( TM \) defined by
\[ Z = X^C + t P + Y^\# = (X^r - \nabla^r Y_s u^s)\partial^r_p + (Y^r + S^r_s u^s)\partial^r_p \]
is a Killing vector field on \((TM,g^S)\). Conversely, any Killing vector field on \((TM,g^S)\) is of this form.
A similar theorem holds for \((TM, g^{CG})\), ([3]). However, in virtue of Lemma 3.5 and the remark after it, the \(Y\) component vanishes.

We shall give a simple sufficient condition for \(\iota P\) to be a Killing vector field on \(TM\). The rest of the section is devoted to investigations on the properties of the \(Y\) component.

Notice that \(a \neq 0\) and \(a_2 = 0\) require \(a_1A \neq 0\). From (3.8) we get immediately

\[(4.28)\quad a_1(\nabla_k \nabla_l Y_{pl} + \nabla_l \nabla_k Y_{pl}) = 2A' g_{kl} Y_{pl} + B(Y_k g_{lp} + Y_l g_{kp}).\]

Since \(b_2 = 0\), symmetrizing (equation II, [9]) in \((k, p)\) we get \(AE_{ikp} = a_2'(g_{ik} Y_{p} + g_{ip} Y_k)\). Consequently, in virtue of the properties of the Lie derivative (2.2), (3.11) and (1.9) yield

\[a_1 \nabla_l P_{kp} = a_2'(g_{kp} Y_k - g_{k} Y_p).\]

Moreover, because of \(a_2 = 0\), \(b_2 = 0\), \(\nabla_l X_q + S_{lq} = P_q = -Pql\), identity (equation I, [9]) together with (3.4) yields

\[BP_{kp} = a_2' \nabla_k Y_p,\]

whence, since \(P\) is skew-symmetric,

\[(4.29)\quad a_2'L_Y g = 0 \text{ and } a_2'Tr(\nabla Y) = 0\]

result.

Next, Lemma 3.9 yields

\[B \nabla_k X_m - a_2' \nabla_k Y_m + BP_{km} = 0,\]

whence we find

\[B \nabla X = 0.\]

We conclude with

**Lemma 4.8.** Suppose (3.1), \(dim M > 2\), and \(a_2 = 0\), \(b_2 = 0\) on \(M \times \{0\}\). If \(a_2' = 0\) on \(M \times \{0\}\), then \(BP = 0\) and \(\nabla P = 0\) on \(M \times \{0\}\).

By Proposition 5.4 we obtain

**Theorem 4.9.** Suppose \(a_2(r^2) = 0\), \(b_2(r^2) = 0\) and \(B(r^2) = 0\) on \((TM, G)\). If \(M\) admits non-trivial skew-symmetric and parallel \((0, 2)\) tensor field \(P\), then its \(\iota\)-lift is a Killing vector field on \(TM\).

**Lemma 4.9.** If \(a_2 = 0\), \(b_2 = 0\) at \((x, 0)\) and \(a \neq 0\) everywhere on \(TM\), then

\[(4.30)\quad 3a_1^2 \left(\nabla^t Y_{p} R_{qtkp} + \nabla^t Y_{p} R_{qtkb} + \nabla^t Y_{p} R_{qtkq} + \nabla^t Y_{p} R_{qtkl} \right) = a_1B \left[ 2\nabla_q Y_k - \nabla_k Y_q \right] g_{ql} + \left(2\nabla_p Y_k - \nabla_k Y_p \right) g_{ql} - \left(\nabla_p Y_q + \nabla_q Y_p \right) g_{kl} \]

\[2A(b_1 - a_1') \left(2\nabla_k Y_{p} g_{pq} - \nabla_l Y_{p} g_{kp} - \nabla_l Y_{q} g_{kp} \right)
\]

and

\[3a_1^2 \nabla Y_{p} R_{qtkp} u^p u^k = 2A(b_1 - a_1') \left[ Y_{k, r^2 - Y_{p, t} u^k u^p} \right] + a_1B \left[ 2Y_{k, x} - Y_{p, k} \right] u_t - g_{kl} Y_{p, q} u^q \]

hold at arbitrary point \((x, 0) \in TM\).

**Proof.** To prove the lemma it is enough to put \(a_2 = 0\), \(b_2 = 0\) in (3.12), then multiply by \(A\) and apply (1.9). For convenience indices \((k, m)\) are interchanged after that. \(\square\)
Lemma 4.10. Suppose (3.1), \( \dim M > 2 \). If neither \( \nabla_n Y_m = 0 \) nor \( \nabla_n Y_m = \frac{T}{n} g_{mn} \), then

\[
\nabla' Y_q R_{rlkp} + \nabla' Y_p R_{rlkq} = 0
\]

if and only if \( B = b = 0 \).

Proof. The "only if " part is obvious. Put \( B \) if and only if

\[
\text{Contracting the right hand side of (4.30) in turn with } g^{kl}, g^{km}, g^{lm} \text{ and } g^{kn}, \text{ we get respectively}
\]

\[
[2Ab + (n - 1)a_1 B] (Y_{m,n} + Y_{n,m}) - 4AbT g_{mn} = 0,
\]

\[
(n - 1) (2Ab - a_1 B) T = 0,
\]

\[-[4Ab + (2n + 1)a_1 B] Y_{k,n} + [2Ab + (n + 2)a_1 B] Y_{n,k} + 2AbT g_{kn} = 0,
\]

\[-a_1 BY_{l,m} + 2[(n - 1)Ab + a_1 B] Y_{m,l} - a_1 BT g_{lm} = 0.
\]

If \( Y_{l,m} - Y_{m,l} = 0 \), then alternating the last two equations in indices we obtain

\[
2Ab + (n + 1)a_1 B = 0,
\]

\[
2(n - 1)Ab + 3a_1 B = 0,
\]

whence \( B = b = 0 \) for \( n \neq 2 \) results.

If \( Y_{l,m} - Y_{m,l} \neq 0 \), then the suitable linear combination of these equations gives

\[
(n - 2) (2Ab - a_1 B) Y_{l,m} + (2Ab - a_1 B) T g_{lm} = 0.
\]

By the second equation this yields \( (2Ab - a_1 B) Y_{l,m} = 0 \). Applying the last result to the first equality completes the proof. \( \square \)

Lemma 4.11. Let \( (TM, G) \) be a tangent bundle of a manifold \( (M, g) \), \( \dim M > 2 \), with non-degenerate \( g \)-natural metric \( G \) given by (2.3). Suppose there is given a Killing vector field \( Z \) on TM with Taylor expansion (3.2) and (3.3). If the coefficients \( a_2(t) \), \( b_2(t) \) vanish along \( M \), then \( Y \) satisfies

\[
A'(2b_1 + a_1') Y = 0,
\]

\[
\{2B(B + A') - 3AB'\} a_1 + AB(2b_1 + a_1') Y = 0.
\]

Proof. Recall that if \( a_2(r_0^2) = 0 \), then necessary \( a_1 A \neq 0 \) on some neighbourhood of \( r_0^2 \).

From (3.9) we easily get

\[
A(K_{ab,c} + K_{bc,a} + K_{ca,b}) + 2(B + A')(g_{bc}Y_a + g_{ca}Y_b + g_{ab}Y_c) = 0.
\]

From Lemma 3.7, by the use of the assumptions on \( a_2 \) and \( b_2 \), we find

\[
2B[K_{ab,(k)l}c + K_{bc,(k)l}a + K_{ca,(k)l}b] + 3B[g_{c(l}T_{k)ab} + g_{a(l}T_{k)bc} + g_{b(l}T_{k)ca}] + 6A'g_{k(}M_{abc} - b[g_{ab}(Y_{c,kl} + Y_{c,lk}) + g_{bc}(Y_{a,kl} + Y_{a,tk}) + g_{ca}(Y_{b,kl} + Y_{b,tk})] + 6B'(g_{a(}g_{kb} + g_{ak}g_{b)}Y_c + (g_{a(}g_{kc} + g_{ak}g_{c)}Y_a + (g_{a(}g_{kc} + g_{ak}g_{c)}Y_b) = 0.
\]
Applying (3.8), (3.6) and (3.7) we find

\[(4.35)\quad 2B \left[ \mathcal{K}_{ab, (k) t} + \mathcal{K}_{bc, (k) t} + \mathcal{K}_{ca, (k) t} \right] - \frac{4B}{a_1} \left[ (g_{a1}g_{bc} + g_{b1}g_{ca} + g_{c1}g_{ab}) Y_k + (g_{a1}g_{bc} + g_{b1}g_{ca} + g_{c1}g_{ab}) Y_t \right] - \frac{4A'(2b_1 + a'_1)}{a_1} (g_{ab} Y_c + g_{bc} Y_a + g_{ca} Y_b) g_{kl} + \frac{6(B' a_1 - B a'_1)}{a_1} \times \left[(g_{a1}g_{kb} + g_{k1}g_{kb}) Y_c + (g_{a1}g_{kc} + g_{k1}g_{kc}) Y_a + (g_{a1}g_{kc} + g_{k1}g_{kc}) Y_b \right] = 0.\]

Hence, contracting with \(g^{il}\), we obtain

\[(4.36)\quad B(\mathcal{K}_{ab, c} + \mathcal{K}_{bc, a} + \mathcal{K}_{ca, b}) + \left[ 3B' - \frac{(B + nA')(2b_1 + a'_1)}{a_1} \right] (g_{bc} Y_a + g_{ca} Y_b + g_{ab} Y_c) = 0.\]

If \(B \neq 0\), then a linear combination of (4.34) and (4.36) yields \(\psi Y = 0\) where

\[(4.37)\quad \psi = 2B (B + A') - 3AB' + \frac{A(B + nA')(2b_1 + a'_1)}{a_1}.\]

On the other hand, contractions of (4.35) with \(g^{ak}\) and then with \(g^{bl}\) yield respectively

\[B \left[(n + 3) \mathcal{K}_{bc, l} + \mathcal{K}_{c, r} g_{bl} + \mathcal{K}_{b, c} g_{cl} \right] = \frac{2}{a_1} [(n + 3) bB + A' (2b_1 + a'_1)] g_{bc} Y_l + \frac{1}{a_1} [2bB + 3(n + 2)(B a'_1 - B' a_1) + 2A' (2b_1 + a'_1)] (g_{a1} Y_c + g_{a1} Y_b) \]

and

\[2B \mathcal{K}_{c, r} = \frac{1}{a_1} [4bB + 3(n + 1)(B a'_1 - B' a_1) + 2A' (2b_1 + a'_1)] Y_c.\]

Hence we find

\[(4.38)\quad 2(n + 3) a_1 B \mathcal{K}_{bc, l} = 4 [(n + 3) bB + A' (2b_1 + a'_1)] g_{bc} Y_l + [3(n + 3)(B a'_1 - B' a_1) + 2A' (2b_1 + a'_1)] (g_{a1} Y_c + g_{a1} Y_b) \]

and

\[(n + 3) a_1 B \left[ \mathcal{K}_{bc, l} + \mathcal{K}_{cl, b} + \mathcal{K}_{bl, c} \right] - [(n + 3) (B (2b_1 + a'_1) - 3a_1 B') + 4A' (2b_1 + a'_1)] (g_{bc} Y_l + g_{cl} Y_b + g_{bl} Y_c) = 0\]

If \(B \neq 0\), then combining the last relation with (4.36) we obtain (4.32) and, as a consequence of (4.37), equality (4.33). On the other hand, if \(B(r^2_0) = 0\), then contractions of (4.38) with \(g^{bc}\) and \(g^{bl}\) yield either \(Y^a = 0\) or \(B' = 0\) and \(A'(2b_1 + a'_1) = 0\). This completes the proof. \(\square\)

**Lemma 4.12.** For an arbitrary \(B\) we have

\[a_1^2 B(\nabla_i \nabla_i Y_k + \nabla_i \nabla_i Y_t) = - 2AB g_{bc} Y_l - \frac{3}{2} A (B a'_1 - a_1 B') (g_{a1} Y_c + g_{a1} Y_b) + \frac{6}{a_1} (B' a_1 - B a'_1) \left[(g_{a1}g_{kb} + g_{k1}g_{kb}) Y_c + (g_{a1}g_{kc} + g_{k1}g_{kc}) Y_a + (g_{a1}g_{kc} + g_{k1}g_{kc}) Y_b \right].\]
By the use of (3.8) and (4.32), we get the second one. On the other hand, if we combine this with (4.38), by the use of (4.32), we find the first equality. Hence, then the previous lemma yields

\[ a_1 A_2 \]

Lemma 4.13. Under hypothesis (3.1) suppose \( dim M > 2 \) and \( a_2 = 0, b_2 = 0 \) on \( M \times \{0\} \subset TM \). Then

(4.39) \[ [Aa'_2(b_1 + a'_1) - 2a_1(Ba'_2 + Ab'_2)] \text{Y} = 0, \]

(4.40) \[ Y \left[ a_1 a'_2 R - \frac{(Ba'_2 + 2Ab'_2)}{2} g \wedge g \right] = 0. \]

If \( a'_2 \neq 0 \), then

(4.41) \[ b'_2 \nabla \text{Y} = 0, \quad (b_1 - a'_1) \nabla \text{Y} = 0. \]

Proof. For the proof of the first part we apply Lemma 3.8. Substituting \( a_2 = 0, b_2 = 0 \), by the use of (1.9), we get

\[ a_1 \left[ 2E_{ab}^p R_{plck} - E_{bk}^p R_{plac} + E_{bc}^p R_{plab} - E_{ak}^p R_{blpc} + E_{ac}^p R_{blpk} \right] + \]

(4.39) \[ \frac{1}{2} [4A' a_1 B + 3A(Ba'_1 - a_1 B')] g_d Y_c - \frac{1}{2} \left[ 2a_1 B^2 + 3A(Ba'_1 - a_1 B') \right] g_d Y_b. \]

This completes the proof. \( \square \)

Applying (3.5) - (3.7) and the Bianchi identity we find

\[ - a_1^2 a'_2 [3R_{blck} Y_a + 3R_{alck} Y_b + (R_{akbl} + R_{bkal}) Y_c - (R_{aclb} + R_{bcal}) Y_k] - \]

(4.39) \[ 2a_1^2 [A(b_1 + a'_1) - a_1 B] g_{ab} (g_{kl} Y_c - g_{cd} Y_k) - \]

(4.39) \[ 2a_1^2 [a_1 (2Bb'_2 - Ba'_2)] [g_{bc} g_{kl} - g_{bk} g_{cl}] Y_a + (g_{ac} g_{kl} - g_{ak} g_{cl}) Y_b + \]

(4.39) \[ a_1 (Ba'_2 + 2Ab'_2) [g_{bk} (g_{ac} Y_c - g_{ac} Y_k) + g_{al} (g_{bc} Y_b - g_{bc} Y_k)] = 0. \]

Symmetrizing the last relation in \( (a, b, l) \) we obtain (4.39). Then, symmetrize in \( (a, b, l) \). Since coefficient times \( Y_c \) vanishes by (4.39), by the use of the Walker lemma and (4.39) we get either \( Y = 0 \) or

\[ a_1 a'_2 (R_{acbl} + R_{albc}) = (Ba'_2 + 2Ab'_2) (g_{al} g_{bc} + g_{ac} g_{bl} - 2g_{ab} g_{cl}), \]

whence, alternating in \( (b, l) \), we easily obtain

\[ a_1 a'_2 R_{acbl} = (Ba'_2 + 2Ab'_2) (g_{al} g_{bc} - g_{ab} g_{cl}). \]

Thus (4.40) is proved.
Suppose now $Y \neq 0$ and $a'_2 \neq 0$ on $M \times \{0\}$. Applying (4.40) to (4.30) and eliminating $B$, by the use of (4.29), we obtain

\begin{align*}
(4.42) \quad 2(b_1 - a'_1) (Y_{n,l}g_{km} + Y_{m,l}g_{kn} - 2Y_k,lg_{mn}) + \\
\left( \frac{2a_1 b'_2}{a_2} - b_1 - a'_1 \right) (Y_{k,m}g_{ln} + Y_{k,n}g_{lm}) - \\
\left( \frac{4a_1 b'_2}{a_2} + b_1 + a'_1 \right) (Y_{m,k}g_{ln} + Y_{n,k}g_{lm}) = 0.
\end{align*}

Contracting (4.42) with $g^{lm}$, by the use of (4.29) we get

\begin{align*}
\left[ (n+1) \frac{a_1 b'_2}{a_2} - (b_1 - a'_1) \right] Y_{k,n} = 0.
\end{align*}

On the other hand, by contraction with $g^{mn}$ we obtain

\begin{align*}
\left[ \frac{a_1 b'_2}{a_2} - (n-1)(b_1 - a'_1) \right] Y_{k,l} = 0.
\end{align*}

Hence we easily get either $\nabla Y = 0$ or both $b'_2 = 0$ and $b_1 - a'_1 = 0$ on $M \times \{0\}$.

Remark 4.2. If $a'_2 \neq 0$ and $Y \neq 0$, then equations (4.41) give a further restriction on the metric $G$. Namely, if $\nabla Y = 0$, then from (1.9) we infer $K = 0$ while from (4.28) we get $A' = 0$ and $B = 0$ on $M \times \{0\}$. Consequently, (4.33) yields $B' = 0$.

Remark 4.3. On the other hand, substituting $b'_2 = 0$ and $b_1 = a'_1$ into Lemma 3.5 we get $b'_1 = 0$ on $M \times \{0\}$.

5. Lifts properties

5.1. Vertical lift $X^v$.

Proposition 5.1. The vertical lift $X^v = X^r \partial_x^r$ of a Killing vector field $X = X^r \partial_x^r$ to $(TM, G)$ with non-degenerate $g$–natural metrics $G$ is a Killing vector field on $TM$ if and only if $a'_j = 0$ and $b_j = 0$ on $TM$.

Proof. Suppose $X^v$ is a Killing vector field. Since $X$ is also the Killing one, ([9], equation 6) yields

\begin{align*}
b_2(X_{r,k}u_l + X_{r,l}u_k)u^r + B(u_k X_l + u_l X_k) + 2u^r X_r(A' g_{kl} + B' u_k u_l) = 0,
\end{align*}

whence, by contraction with $g^{kl}$ and $u^k u^l$ we obtain

\begin{align*}
2u^r X_r (B + nA' + r^2 B') = 0
\end{align*}

and

\begin{align*}
2r^2 u^r X_r (B + A' + r^2 B') = 0
\end{align*}

since $X$ is a Killing vector field on $M$. Thus $A' = 0$ and the only smooth solution to $B + r^2 B' = 0$ on $TM$ is $B = 0$. In similar manner, from ([9], equation 7 and 8) we deduce that $a'_1 = a'_2 = 0$ and $b_1 = b_2 = 0$ on $TM$. The "only if" part is obvious. Thus the proposition is proved. \qed
5.2. $V^a\partial^p_a = u^p\nabla^TY_p\partial^p_a$. Let $Y$ be a non-parallel Killing vector field on $M$ and consider its lift $u^p\nabla^TY_p\partial^p_a$ to $(TM,G)$. Then we have $\partial^p_aV^a = \nabla^aY_k, \partial^k_aV^a = u^p\Theta_k (\nabla^aY_p)$ and from ([9] or [10], equations 6, 7 and 8) we obtain

$$(L_{V^a\partial^p_a}G) (\partial^p_a, \partial^k_l) = a_2(\nabla_k\nabla_l\nabla_{p}\nabla_p + \nabla_l\nabla_{p}\nabla_p) + B(\nabla_k\nabla_l\nabla_p + \nabla_p + \nabla_l\nabla_p),$$

$$(L_{V^a\partial^p_a}G) (\partial^p_a, \partial^k_l) = a_2\nabla_k\nabla_{p}\nabla_p + b_2\nabla_l\nabla_p + b_2\nabla_l\nabla_p,$$

$$(L_{V^a\partial^p_a}G) (\partial^p_a, \partial^k_l) = 0.$$  

Hence we deduce

**Proposition 5.2.** Let $Y$ be a non-parallel Killing vector field on $M$ satisfying $\nabla^aY_p = 0$. Then $u^p\nabla^TY_p\partial^p_a$ is a Killing vector field on $TM$ if and only if $a_2 = b_2 = 0 \iff \nabla^aY_p = 0$ on $TM$.

**Proposition 5.3.** Let $Y$ be a non-parallel Killing vector field on $M$. If $a_2 = b_2 = 0 \iff \nabla^aY_p = 0$ on $TM$ and $u^p\nabla^TY_p\partial^p_a$ is a Killing vector field on $TM$ then $\nabla^aY_p = 0$ on $M$.

5.3. $\iota P$.

**Proposition 5.4.** Let $P$ be an arbitrary $(0,2)$-tensor field on $(M,g)$. Then its $\iota$ - lift $\iota P = u^aP^a\partial_a$ to $(TM,G)$ with non-degenerate $g$- natural metric $G$ satisfies

$$(L_{\iota P}G) (\partial^a_k, \partial^a_l) = a_2u^a(\nabla_kP_{kr} + \nabla_lP_{kr}) + b_2u^a(\nabla_kP_{pr} + \nabla_lP_{pr}u_k) + 2(a_1\nabla_k + \nabla_l)P_{pr}u^a + b_2u^a(\nabla_kP_{pr}u_k + \nabla_lP_{pr}u_k),$$

$$(L_{\iota P}G) (\partial^a_k, \partial^a_l) = a_2\nabla_kP_{pr} + b_2\nabla_lP_{pr} + a_1u^a\nabla_kP_{pr} + b_2\nabla_lP_{pr}u_k + 2(a_1\nabla_k + \nabla_l)P_{pr}u^a + b_2u^a(\nabla_kP_{pr}u_k + \nabla_lP_{pr}u_k),$$

$$(L_{\iota P}G) (\partial^a_k, \partial^a_l) = a_1(P_{kl} + P_{lk}) + b_1[u^a(P_{rp} + P_{pk})u_l + u^a(P_{rp} + P_{pl})u_k] + 2(a_1\nabla_k + \nabla_l)\nabla^aP_{pr}u^a.$$

**Proof.** The Proposition follows from ([9] or [10], equations 6, 7 and 8), where we have $H^a = 0, V^a = u^aP^a_{pr}, \partial^a_kV^a = P^a_{kr}, \partial^a_kV^a = u^a(\partial^a_kP^a_{pr} - \Gamma^a_{pr}P^a_{tk})$ and $\partial^a_kV^a + V^a\nabla^a_{kr} = u^a\nabla^a_{kr}P^a_{pr}$. Hence we easily get

**Proposition 5.5.** Let $P$ be a skew-symmetric $(0,2)$-tensor field on $(M,g)$. Then its $\iota$ - lift $\iota P = u^aP^a\partial_a$ to $(TM,G)$ with non-degenerate $g$- natural metric $G$ satisfies

$$(L_{\iota P}G) (\partial^a_k, \partial^a_l) = a_2(u^a\nabla_kP_{kr} + u^a\nabla_lP_{kr}) + B(u^aP_{pr}u_k + u^aP_{pr}u_k),$$

$$(L_{\iota P}G) (\partial^a_k, \partial^a_l) = a_2\nabla_kP_{pr} + b_2\nabla_lP_{pr} + a_1u^a\nabla_kP_{pr},$$

$$(L_{\iota P}G) (\partial^a_k, \partial^a_l) = 0.$$  

5.3.1. $\iota C^{[X]}$. Put $C^{[X]} = \left( (C^{[X]})^h_k \right) = (-g^{hr}(L_Xg)_{rk}) = (-\nabla^hX_k + \nabla_kX^h)$ on $(M,g)$. Then its $\iota$-lift $\iota C^{[X]} = \left( 0, u^k (C^{[X]})^h_k \right) = \left( 0, -u^k(\nabla^hX_k + \nabla_kX^h) \right)$ is a vertical vector field on $TM$. In adapted coordinates $(\partial^a_k, \partial^a_l)$ we have

$$\iota C^{[X]} = -u^k(\nabla^hX_k + \nabla_kX^h) \partial^a_k.$$
Applying ([9] or [10], equations 6, 7 and 8), we easily get

\[(L_{C\times X}(G) (\partial_k^n, \partial_k^b) =
\]
\[-a_2 u^p \left[ \nabla_k \nabla_1 X_p + \nabla_1 \nabla_k X_p + \nabla_k \nabla_p X_1 + \nabla_1 \nabla_p X_k \right] -
2b_2 u^p u^q \left[ \nabla_k \nabla_1 \nabla_p X_q u_l + \nabla_l \nabla_1 \nabla_p X_q u_k \right] - 4 \left( A' g_{kl} + B' u_k u_l \right) u^p u^q \nabla_p X_q -
Bu^p \left[ (\nabla_k X_p + \nabla_p X_k) u_l + (\nabla_1 X_p + \nabla_p X_1) u_k \right],
\]

\[(L_{C\times X}(G) (\partial_k^l, \partial_k^b) =
\]
\[-a_2 \left( \nabla_k X_l + \nabla_l X_k \right) - b_2 u^p \left[ 2 \left( \nabla_k X_p + \nabla_p X_k \right) u_l + \left( \nabla_1 X_p + \nabla_p X_1 \right) u_k \right] -
a_1 u^p \left( \nabla_1 \nabla_k X_p + \nabla_k \nabla_p X_1 \right) - 2b_1 u^p u^q \nabla_k \nabla_p X_q u_k -
4 \left( a_2' g_{kl} + b_2' u_k u_l \right) u^p u^q \nabla_p X_q,
\]

\[(L_{C\times X}(G) (\partial_k^l, \partial_k^n) =
\]
\[-2a_1 \left( \nabla_k X_l + \nabla_l X_k \right) - 2b_1 u^p \left[ \left( \nabla_k X_p + \nabla_p X_k \right) u_l + \left( \nabla_1 X_p + \nabla_p X_1 \right) u_k \right] -
4 \left( a_1' g_{kl} + b_1' u_k u_l \right) u^p u^q \nabla_p X_q.
\]

5.3.2. Complete lift $X^C$ of $X$ to $(TM, G)$. We have $X^C = \left( X^C \partial_r \right)^C = X^C \partial_r + \partial X^C \delta_r = X^C \partial_r + u^p \nabla_p X^C \partial^r$. Making use of ([9] or [10], equations 6, 7 and 8) we obtain

\[(L_{C\times X}(G) (\partial_k^n, \partial_k^b) =
\]
\[a_2 u^p \left[ \nabla_k \nabla_p X_1 + X^r R_{rkpl} \right] + \nabla_l \nabla_p X_k + X^r R_{rlpk} \right] +
2b_2 u^p u^q \left[ \left( \nabla_k \nabla_p X_q + X^r R_{rqlp} \right) u_l + \left( \nabla_l \nabla_p X_q + X^r R_{rlpq} \right) u_k \right] +
A \left( \nabla_k X_l + \nabla_l X_k \right) + Bu^p \left[ \left( \nabla_k X_p + \nabla_p X_k \right) u_l + \left( \nabla_1 X_p + \nabla_p X_1 \right) u_k \right] +
2 \left( A' g_{kl} + B' u_k u_l \right) u^p u^q \nabla_p X_q,
\]

\[(L_{C\times X}(G) (\partial_k^l, \partial_k^b) =
\]
\[a_1 u^p \left[ \nabla_1 \nabla_p X_k + X^r R_{rlpk} \right] + a_2 \left( \nabla_k X_l + \nabla_l X_k \right) +
b_2 u^p \left[ \left( \nabla_k X_p + \nabla_p X_k \right) u_l + \left( \nabla_1 X_p + \nabla_p X_1 \right) u_k \right] +
2b_1 u^p u^q \left( \nabla_k \nabla_p X_q + X^r R_{rqlp} \right) u_k + 2 \left( a_2' g_{kl} + b_2' u_k u_l \right) u^p u^q \nabla_p X_q,
\]

\[(L_{C\times X}(G) (\partial_k^l, \partial_k^n) =
\]
\[a_1 \left( \nabla_k X_l + \nabla_l X_k \right) + b_1 u^p \left[ \left( \nabla_k X_p + \nabla_p X_k \right) u_l + \left( \nabla_1 X_p + \nabla_p X_1 \right) u_k \right] +
2 \left( a_1' g_{kl} + b_1' u_k u_l \right) u^p u^q \nabla_p X_q.
\]

5.3.3. $t^{C[X]} + X^C$ for an infinitesimal affine transformation. Suppose that $X$ is an infinitesimal affine transformation on $M$. Then by (2.2) and the definition

\[\nabla_k \nabla_1 X_p + \nabla_k \nabla_p X_1 = \nabla_k \nabla_1 X_p + X^r R_{rkpl} + \nabla_k \nabla_p X_1 + X^r X_r \nabla_1 X_p \]

and

\[u^p u^q \nabla_k \nabla_p X_q = -u^p u^q X^r R_{rqlp} = 0,
\]

Therefore, applying results of previous subsections, we find

\[(L_{C\times X}(G) (\partial_k^n, \partial_k^b) = A \left( \nabla_k X_l + \nabla_l X_k \right) - 2 \left( A' g_{kl} + B' u_k u_l \right) u^p u^q \nabla_p X_q,
\]
\( (L_{\xi C} + X \cdot G) \left( \partial_k^c, \partial_k^b \right) = \)
\[-2 \left( a'_2 g_{kl} + b'_2 u_k u_l \right) u^p u^q \nabla_p X_q - b_2 u^p (\nabla_k X_p + \nabla_p X_k) u_l, \]
\[= -2 \left( a'_1 g_{kl} + b'_1 u_k u_l \right) u^p u^q \nabla_p X_q. \]

\[ (L_{\xi C} + X \cdot G) \left( \partial_k^c, \partial_k^b \right) = -a_1 \left( \nabla_k X_l + \nabla_l X_k \right) - b_1 u^p \left[ (\nabla_k X_p + \nabla_p X_k) u_l + (\nabla_l X_p + \nabla_p X_l) u_k \right] - 2 \left( a'_1 g_{kl} + b'_1 u_k u_l \right) u^p u^q \nabla_p X_q. \]

6. Appendix

A \((0, 4)\) tensor \(B\) on a manifold \(M\) is said to be a generalized curvature tensor if
\[ B(V, X, Y, Z) + B(V, Y, Z, X) + B(V, Z, X, Y) = 0 \]
and
\[ B(V, X, Y, Z) = -B(X, V, Y, Z), \quad B(V, X, Y, Z) = B(Y, Z, V, X) \]
for all vector fields \(V, X, Y, Z\) on \(M\) ([16]).

For a \((0, k)\) tensor \(T\), \(k \geq 1\), we define
\[(R \cdot T)(X_1, ..., X_k; X, Y) = \nabla_Y \nabla_X T(X_1, ..., X_k) - \nabla_X \nabla_Y T(X_1, ..., X_k).\]

For more details see for example ([6]) or ([11]).

The Kulkarni-Nomizu product of symmetric \((0, 2)\) tensors \(A\) and \(B\) is given by

**Theorem 6.1.** [12] Let \((M, g)\) be a semi-Riemannian manifold with metric \(g\), \(\text{dim}M > 2\). Let \(g_X\) be a 1-form associated to \(g\), i.e. \(g_X(Y) = g(Y, X)\) for any vector field \(Y\).

If \(B\) is generalized curvature tensor having the property \(R \cdot B = 0\) and \(P\) is a one-form on \(M\) satisfying
\[(6.1) \quad (R \cdot V)(X; Y, Z) = (P \wedge g_X)(Y, Z), \]
for some 1-form \(V\), then
\[P \left( B - \frac{\text{Tr} B}{2n(n - 1)} g \wedge g \right) = 0. \]

If \(A\) is a symmetric \((0, 2)\)-tensor on \(M\) having the properties \(R \cdot A = 0\) and (6.1) then
\[P \left( A - \frac{\text{Tr} A}{n} g \right) = 0. \]

**Lemma 6.1.** [19] Let \(A_l, B_{hk}\) where \(l, h, k = 1, ..., n\) be numbers satisfying
\[B_{hk} = B_{kh}, \quad A_l B_{hk} + A_h B_{kl} + A_k B_{lh} = 0.\]

Then either \(A_l = 0\) for all \(l\) or \(B_{hk} = 0\) for all \(h, k\).
References

[12] Grycak, W., On generalized curvature tensors and symmetric (0,2)-tensors with symmetry condition imposed on the second derivative, Tensor N.S., 33 no. 2, (1979), 150-152.

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