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Chen Inequalities on Lightlike Hypersurface of a Lorentzian Manifold with Semi-Symmetric Metric Connection

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ABSTRACT

In this paper, we introduce k-Ricci curvature and k-scalar curvature on lightlike hypersurface of a Lorentzian manifold with semi-symmetric metric connection. Using this curvatures, we establish some inequalities for lightlike hypersurface of a Lorentzian manifold with semi-symmetric metric connection. Considering these inequalities, we obtain the relation between Ricci curvature and scalar curvature endowed with semi-symmetric metric connection.

Keywords: *Chen inequality; lightlike hypersurface; Lorentzian manifold; semi-symmetric metric connection.*
AMS Subject Classification (2010): *Primary: 53B05; 53B15; 53C40; 53C42; 53C50.*

1. Introduction

Hayden [17] introduced a semi-symmetric metric connection on a Riemannian manifold. Imai [20] gave basic properties of a hypersurface of a Riemannian manifold with semi-symmetric metric connection and get conformal equations of Gauss and Codazzi. Konar and Biswas [22] considered semi-symmetric metric connection on Lorentz manifold. They showed that the perfect fluid space time with a non-zero constant scalar curvature which admits a semi-symmetric metric connection whose Ricci tensor is zero has vanishing expansion scalar and acceleration vector.

In 1993, Chen [9] introduced a new Riemannian invariant for a Riemannian manifold M as follows:

$$\delta_M = \tau(p) - inf(K)(p) \tag{1.1}$$

where $\tau(p)$ is scalar curvature of M and

$$inf(K)(p) = \{inf K(\Pi) : K(\Pi) \ \ is \ \ a \ \ plane \ \ section \ \ of \ \ T_pM\}.$$

In [5], Chen established a sharp inequality for submanifold in a real space form involving intrinsic invariants, namely the sectional curvature and the scalar curvature of the submanifold; and the main extrinsic invariant, namely the squared mean curvature.

Afterwards, Chen and some geometers studied similar problems for non-degenerate submanifolds of different spaces such as in [4, 6, 8, 26]. Later Mihai and Özgür in [23] studied Chen inequalities on submanifolds of real space forms endowed with semi-symmetric metric connection.

Gülbahar, Kılıç and Keleş introduced Chen-like inequalities and curvature invariants in lightlike geometry. Also, they established some inequalities between the extrinsic scalar curvatures and the intrinsic scalar curvatures [14]. In [15], they established some inequalities involving k-Ricci curvature, k-scalar curvature, the screen scalar curvature on a screen homothetic lightlike hypersurface of a Lorentzian manifold. Poyraz and Yaşar introduced k-Ricci curvature and k-scalar curvature on lightlike hypersurface of a Lorentzian product manifold with quarter-symmetric nonmetric connection and using these curvatures they established some

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Chen-type inequalities for screen homothetic lightlike hypersurface of a Lorentzian product manifold with quarter-symmetric nonmetric connection [25].

In this paper, we study inequalities for screen homothetic lightlike hypersurface of a real space form $\tilde{M}(c)$ of constant sectional curvature c, endowed with semi symmetric metric connection. Considering these inequalities, we obtain the relation between Ricci curvature and scalar curvature endowed with the semi symmetric metric connection.

2. Preliminaries

Let M be a hypersurface of a $(n+1)-$dimensional, $n>1$, semi-Riemannian manifold $\widetilde{M}$ with semi-Riemannian metric $\widetilde{g}$ of index $1\leq\nu\leq n$. We consider

$$T_xM^{\perp}=\left\{Y_x\in T_x\widetilde{M}\mid \widetilde{g}_x\left(Y_x,X_x\right)=0,\forall X_x\in T_xM\right\}$$

for any $x\in M$. Then we say that M is a *lightlike (null, degenerate) hypersurface* of $\widetilde{M}$ or equivalently, the immersion

$$i:M\to\widetilde{M}$$

of M in $\widetilde{M}$ is *lightlike (null, degenerate)* if $T_xM\cap T_xM^{\perp}\neq\{0\}$ at any $x\in M$.

An orthogonal complementary vector bundle of $TM^{\perp}$ in TM is non-degenerate subbundle of TM named the *screen distribution* on M and denoted $S\left(TM\right)$. We have the following splitting into orthogonal direct sum:

$$TM=S\left(TM\right)\perp TM^{\perp}. \tag{2.1}$$

The subbundle $S\left(TM\right)$ is non-degenerate, so is $S\left(TM\right)^{\perp}$, and the following satisfies:

$$T\widetilde{M}=S\left(TM\right)\perp S\left(TM\right)^{\perp}, \tag{2.2}$$

where $S\left(TM\right)^{\perp}$ is the orthogonal complementary vector bundle to $S\left(TM\right)$ in $T\widetilde{M}\Big|_M$.

Let $tr(TM)$ denote the complementary vector bundle of $TM^{\perp}$ in $S\left(TM\right)^{\perp}$. Then we have

$$S\left(TM\right)^{\perp}=TM^{\perp}\oplus tr(TM). \tag{2.3}$$

Let $\mathcal{U}$ be a coordinate neighborhood in M and ξ be a basis of $\Gamma(TM^{\perp}|_{\mathcal{U}})$. Then there exists a basis N of $tr\left(TM\right)|_U$ satisfying the following conditions:

$$\tilde{g}(N,\xi)=1,$$

and

$$\tilde{g}\left(N,N\right)=\tilde{g}(W,N)=0,\quad\forall W\in\Gamma(\,S\left(TM\right)|_{\mathcal{U}}).$$

The subbundle $tr\left(TM\right)$ is named a *lightlike transversal vector bundle* of M. We note that $tr\left(TM\right)$ is never orthogonal to TM. From (2.1), (2.2) and (2.3) we have

$$T\widetilde{M}\Big|_M=S\left(TM\right)\perp\left(TM^{\perp}\oplus tr\left(TM\right)\right)=TM\oplus tr\left(TM\right), \tag{2.4}$$

[11, 16].

3. Semi-Symmetric Metric Connection

For $n>1$, let $\widetilde{M}$ be an $(n+2)-$dimensional differentiable manifold of class C^{∞} and $\widetilde{\nabla}$ a linear connection in $\widetilde{M}$. The torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$ is given by

$$\widetilde{T}(\widetilde{X},\widetilde{Y})=\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}-\widetilde{\nabla}_{\widetilde{Y}}\widetilde{X}-[\widetilde{X},\widetilde{Y}],\ \ \forall\widetilde{X},\widetilde{Y}\in\Gamma(T\widetilde{M})$$

and have type $(1,2)$. When the torsion tensor $\widetilde{T}$ satisfies

$$\widetilde{T}(\widetilde{X},\widetilde{Y})=\widetilde{\pi}(\widetilde{Y})\widetilde{X}-\widetilde{\pi}(\widetilde{X})\widetilde{Y}$$

for a $1-$form $\widetilde{\pi}$, the connection $\widetilde{\nabla}$ is said to be *semi-symmetric* (see [27]).

Let us consider a semi-Riemannian metric $\widetilde{g}$ of index ν with $1\leq\nu\leq n+1$ in $\widetilde{M}$ and $\widetilde{\nabla}$ satisfying

$$\widetilde{\nabla}\widetilde{g}=0.$$

A linear connection of this type is called a *metric connection* (see [23]).

We assume that the semi-Riemannian manifold $\widetilde{M}$ admits a semi-symmetric metric connection which is given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}=\overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y}+\widetilde{\pi}(\widetilde{Y})\widetilde{X}-\widetilde{g}(\widetilde{X},\widetilde{Y})\tilde{Q} \tag{3.1}$$

for arbitrary vector fields $\widetilde{X}$ and $\widetilde{Y}$ of $\widetilde{M}$, where $\overset{\circ}{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the semi-Riemannian metric $\widetilde{g}$, $\widetilde{\pi}$ is a $1-$form and $\tilde{Q}$ is the vector field defined by

$$\widetilde{g}(\tilde{Q},\widetilde{X})=\widetilde{\pi}(\widetilde{X})$$

for an arbitrary vector field $\widetilde{X}$ of $\widetilde{M}$ (see [13] and [27]).

The *Gauss formula* with respect to the induced connection ∇ on the lightlike hypersurface from the semi-symmetric metric connection $\widetilde{\nabla}$ is given by

$$\widetilde{\nabla}_XY=\nabla_XY+m\left(X,Y\right)N \tag{3.2}$$

for arbitrary vector fields X and Y of M, where m is a tensor of type $(0,2)$ of the lightlike hypersurface of M [28].

On the other hand, denoting the projection of TM on $S\left(TM\right)$ with respect to the decomposition (2.1) by P, one has the *Weingarten formula* with respect to the semi-symmetric connection which is given by

$$\nabla_XPY=\overset{*}{\nabla}_XPY+D(X,PY)\xi, \tag{3.3}$$

where $\overset{*}{\nabla}_XPY$ belongs to $\Gamma(S\left(TM\right))$ and D is $1-$form on M.

The curvature tensor $\overset{\circ}{\widetilde{R}}$ with respect to $\overset{\circ}{\widetilde{\nabla}}$ on real space form $\widetilde{M}(c)$ is defined by

$$\overset{\circ}{\widetilde{R}}(X,Y,Z,W)=c\{g(X,W)g(Y,Z)-g(Y,W)g(X,Z)\}. \tag{3.4}$$

Using (3.1), for any vector fields $X,Y,Z,W\in\Gamma(TM)$ and $(0,2)$ tensor field α which defined by

$$\alpha(X,Y)=(\overset{\circ}{\widetilde{\nabla}}_X\pi)Y-\pi(X)\pi(Y)+\frac{1}{2}\pi(Q)g(X,Y) \tag{3.5}$$

we have relation between the curvature tensor $\overset{\circ}{\widetilde{R}}$ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ and the curvature tensor $\widetilde{R}$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ given by

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= \overset{\circ}{\widetilde{R}}(X,Y,Z,W)-\alpha(Y,Z)g(X,W)+\alpha(X,Z)g(Y,W)\\ &\quad -\alpha(X,W)g(Y,Z)+\alpha(Y,W)g(X,Z),\end{aligned} \tag{3.6}$$

[19].

Moreover, Gauss-Codazzi equations with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ on $\widetilde{M}$ can be written as [28]

$$\begin{aligned}R(X,Y,Z,PW) &= \widetilde{R}(X,Y,Z,PW)-m(X,Z)D(Y,PW)+m(Y,Z)D(X,PW)\\ &\quad -\{m(X,Z)\eta(Y)-m(Y,Z)\eta(X)\}\pi(PW),\end{aligned} \tag{3.7}$$

$$\widetilde{g}(\tilde{R}(X,Y)Z,\xi) = \pi(Y)\,m(X,Z) - \pi(X)\,m(Y,Z) + (\nabla_X m)(Y,Z) - (\nabla_Y m)(X,Z) + m(Y,Z)(\tau(X) - \mu\eta(X)) - m(X,Z)(\tau(Y) - \mu\eta(Y)), \quad (3.8)$$

and

$$\widetilde{g}(\tilde{R}(X,Y)Z,N) = g(R(X,Y)Z,N), \quad (3.9)$$

for any vector fields $X,Y,Z,W \in \Gamma(TM)$.

From (3.4), (3.6) and (3.7), we have

$$\begin{aligned} R(X,Y,Z,PW) = {} & c\{g(Y,Z)g(X,PW) - g(X,Z)g(Y,PW)\} \\ & -\alpha(Y,Z)g(X,PW) + \alpha(X,Z)g(Y,PW) \\ & -\alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z) \\ & -m(X,Z)D(Y,PW) + m(Y,Z)D(X,PW) \\ & -\{m(X,Z)\eta(Y) - m(Y,Z)\eta(X)\}\pi(PW). \end{aligned} \quad (3.10)$$

Denote by λ the trace of α.

Let$(M,g,S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\widetilde{M},\widetilde{g})$. Then M is named totally umbilical lightlike hypersurface if there exists a smooth function such that

$$m(X,Y)_p = Hg_p(X,Y), \qquad X,Y \in \Gamma(T_pM) \quad (3.11)$$

for any coordinate neighborhood U and $X,Y \in \Gamma(TM_{|U})$, where $H \in R$. If every points of M is umbilical, the lightlike hypsersurface M is named totally umbilical in $\widetilde{M}$ [11]. If $m = 0$, then the lightlike hypsersurface M is named totally geodesic in $\widetilde{M}$.

The mean curvature μ of M with respect to an orthonormal basis $\{e_1,...,e_n\}$ of $\Gamma(S(TM))$ is defined by

$$\mu = \frac{1}{n}tr(m) = \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i m(e_i,e_i), \qquad g(e_i,e_i) = \varepsilon_i. \quad (3.12)$$

A lightlike hypersurface (M,g) of a semi-Riemannian manifold $(\widetilde{M},\widetilde{g})$ is called *screen locally conformal* if the shape operators A_N and $\overset{*}{A}_\xi$ of M and $S(TM)$, respectively, are related by

$$A_N = \varphi\overset{*}{A}_\xi, \quad (3.13)$$

where φ is a non-vanishing smooth function on a neighborhood $\mathcal{U}$ on M. In particular, , if φ is a non-zero constant, M is called screen homothetic [12].

Let M be a two-dimensional non-degenerate plane. The sectional curvature at $p \in M$ is given by

$$K_{ij} = \frac{g(R(e_j,e_i)e_i,e_j)}{g(e_i,e_i)g(e_j,e_j) - g(e_i,e_j)^2} \quad (3.14)$$

[12].

Let $p \in M$ and ξ be null vector of T_pM. A plane Π of T_pM is said to be null plane if it contains ξ and e_i such that $g(\xi,e_i) = 0$ and $g(e_i,e_i) = \varepsilon_i = \pm 1$. One defines the null sectional curvature of Π by

$$K_i^{null} = \frac{g(R_p(e_i,\xi)\xi,e_i)}{g_p(e_i,e_i)}$$

[2].

We denote the Ricci tensor of $\widetilde{M}$ with $\widetilde{Ric}$ and the induced Ricci type tensor of M with $R^{(0,2)}$. Then, $\widetilde{Ric}$ and $R^{(0,2)}$ are given by

$$\begin{aligned} \widetilde{Ric}(X,Y) &= trace\{Z \to \widetilde{R}(Z,X)Y\}, \;\; \forall X,Y \in \Gamma(T\widetilde{M}), \quad (3.15) \\ R^{(0,2)}(X,Y) &= trace\{Z \to R(Z,X)Y\}, \;\; \forall X,Y \in \Gamma(TM), \end{aligned}$$

where

$$R^{(0,2)}(X,Y) = \sum_{i=1}^{n}\varepsilon_i g(R(e_i,X)Y,e_i) + \tilde{g}(R(\xi,X)Y,N) \quad (3.16)$$

for the quasi-orthonormal frame $\{e_1, ..., e_n, \xi\}$ of T_pM.

Scalar curvature τ is defined

$$\tau(p) = \sum_{i,j=1}^{n} K_{ij} + \sum_{i=1}^{n} K_i^{null} + K_{iN}, \tag{3.17}$$

where $K_{iN} = \tilde{g}(R(\xi, e_i)e_i, N)$ for $i \in \{1, ..., n\}$ [10].

If $dim(M) > 2$ and

$$Ric(X, Y) = kg(X, Y), \tag{3.18}$$

then M is an Einstein manifold. For $dim(M) = 2$, any M is Einstein but k in (3.18) is not necessarily constant [12].

4. Chen Ricci Inequality

In this section, we use the same notations and terminologies as in [14].

Let M be an $(n + 1)$-dimensional lightlike hypersurface of a Lorentzian manifold $\widetilde{M}$ with a semi-symmetric metric connection. $\{e_1, ..., e_n, \xi\}$ and $\{e_1, ..., e_n\}$ are basis of $\Gamma(TM)$ and an orthonormal basis of $\Gamma(S(TM))$, respectively. Similarly, for $k \leq n$, $\pi_{k,\xi} = sp\{e_1, ..., e_k, \xi\}$ and $\pi_k = sp\{e_1, ..., e_k\}$ are $(k + 1)-$dimensional degenerate plane section and $\pi_k = sp\{e_1, ..., e_k\}$ is $k-$dimensional non-degenerate plane section, respectively. For a unit vector $X \in \Gamma(TM)$, the $k-$degenerate Ricci curvature and the $k-$Ricci curvature are defined by

$$Ric_{\pi_{k,\xi}}(X) = R^{(0,2)}(X, X) = \sum_{j=1}^{k} g(R(e_j, X)X, e_j) + \widetilde{g}(R(\xi, X)X, N), \tag{4.1}$$

$$Ric_{\pi_k}(X) = R^{(0,2)}(X, X) = \sum_{j=1}^{k} g(R(e_j, X)X, e_j), \tag{4.2}$$

respectively [14]. Also for $p \in M$, $k-$degenerate scalar curvature and $k-$scalar curvature are determined by

$$\tau_{\pi_{k,\xi}}(p) = \sum_{i,j=1}^{k} K_{ij} + \sum_{i=1}^{k} K_i^{null} + K_{iN}, \tag{4.3}$$

$$\tau_{\pi_k}(p) = \sum_{i,j=1}^{k} K_{ij}, \tag{4.4}$$

respectively [14]. For $k = n$, $\pi_n = sp\{e_1, ..., e_n\} = \Gamma(S(TM))$, we have the screen Ricci curvature and the screen scalar curvature given by

$$Ric_{S(TM)}(e_1) = Ric_{\pi_n}(e_1) = \sum_{j=1}^{n} K_{1j} = K_{12} + ... + K_{1n}, \tag{4.5}$$

and

$$\tau_{S(TM)} = \sum_{i,j=1}^{n} K_{ij}, \tag{4.6}$$

respectively [14].

Using (3.10) we obtain

$$\tau_{S(TM)}(p) = n(n - 1)c - 2(n - 1)\lambda + \sum_{i,j=1}^{n} m_{ii}D_{jj} - m_{ij}D_{ji}, \tag{4.7}$$

where λ is the trace of α and $m_{ij} = m(e_i, e_j)$, $D_{ij} = D(e_i, e_j)$ for $i, j \in \{1, ..., n\}$.

Let $\widetilde{M}(c)$ be a Lorentzian space form and M be a screen homothetic lightlike hypersurface of an $(n+2)$-dimensional $\widetilde{M}(c)$. Using (3.6)-(3.10) we get the following equations:

$$\tau_{S(TM)}(p) = n(n-1)c - 2(n-1)\lambda + \varphi n^2\mu^2 - \varphi \sum_{i,j=1}^{n} (m_{ij})^2, \tag{4.8}$$

$$\begin{aligned} \sum_{i=1}^{n} K_i^{null} &= \sum_{i=1}^{n} R(e_i, \xi, \xi, e_i) \\ &= \sum_{i=1}^{n} \widetilde{R}(e_i, \xi, \xi, e_i) \\ &= \sum_{i=1}^{n} -\alpha(\xi, \xi) = -n\alpha(\xi, \xi), \end{aligned} \tag{4.9}$$

$$\begin{aligned} \sum_{i=1}^{n} K_i^{N} &= \sum_{i=1}^{n} R(\xi, e_i, e_i, N) \\ &= \sum_{i=1}^{n} \widetilde{R}(\xi, e_i, e_i, N) \\ &= \sum_{i=1}^{n} (c - \alpha(\xi, N) - \alpha(e_i, e_i)) \\ &= nc - n\alpha(\xi, N) - \lambda. \end{aligned} \tag{4.10}$$

From (3.17), (4.8), (4.9) and (4.10), we get the induced scalar curvature $\tau(p)$ of M as following:

$$\tau(p) = n^2c - 2(n+1)\lambda + \varphi n^2\mu^2 - \varphi \sum_{i,j=1}^{n} (m_{ij})^2 - n\left(\alpha(\xi, \xi) + \alpha(\xi, N)\right). \tag{4.11}$$

Using (4.11) we obtain the following :

Theorem 4.1. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then we have*

$$\frac{1}{\varphi}\left(\tau(p) - n^2c + 2(n+1)\lambda + n\left(\alpha(\xi, \xi) + \alpha(\xi, N)\right)\right) \leq n^2\mu^2 \tag{4.12}$$

The equality of (4.12) holds for $p \in M$ if and only if p is a totally geodesic point..

Lemma 4.1. *[26] Let $a_1, a_2, ..., a_n$, be n-real number $(n > 1)$, then*

$$\frac{1}{n}(\sum_{i=1}^{n} a_i)^2 \leq \sum_{i=1}^{n} a_i^2$$

with equality iff $a_1 = a_2 = ... = a_n$.

Theorem 4.2. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then we have*

$$\frac{1}{\varphi}\left(\tau(p) - n^2c + 2(n+1)\lambda + n\left(\alpha(\xi, \xi) + \alpha(\xi, N)\right)\right) \leq n(n-1)\mu^2. \tag{4.13}$$

For $p \in M$ the equality of (4.13) satisfies iff p is a totally umbilical point.

Proof. Using Lemma 4.1 one derives

$$n\mu^2 \leq \sum_{i=1}^{n}(m_{ii})^2. \tag{4.14}$$

After substituting (4.14) in (4.11) we find (4.13). For $p \in M$ the equality of (4.13) satisfies iff

$$m_{11} = ... = m_{nn}.$$

Thus p is a totally umbilical point. □

If the sectional curvature is screen homothetic, then the sectional curvature of lightlike hypersurface is symmetric. One defines the screen scalar curvature $r_{S(TM)}$

$$r_{S(TM)}(p) = \sum_{1\leq i<j\leq n} K_{ij} = \frac{1}{2}\sum_{i,j=1}^{n} K_{ij} = \frac{1}{2}\tau_{S(TM)}(p). \tag{4.15}$$

By using (4.8), the equality (4.15) can be rewritten as follows:

$$2r_{S(TM)}(p) = n(n-1)c - 2(n-1)\lambda + \varphi n^2\mu^2 - \sum_{i,j=1}^{n}(m_{ij})^2. \tag{4.16}$$

Theorem 4.3. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ such that the vector field P is tangent to M. Then, the following statements are true.*

(i) For $X \in S^1(TM) = \{X \in S(TM) : \langle X, X\rangle = 1\}$

$$Ric_{S(TM)}(X) \leq \frac{1}{4}\varphi n^2\mu^2 + (n-1)c - (2n-3)\lambda + (n-2)\alpha(X,X). \tag{4.17}$$

(ii) The equality case of (4.17) is satisfied by $X \in T_p^1(M)$ iff

$$\begin{aligned} m(X,Y) &= 0, \text{ for all } Y \in T_p(M) \text{ orthogonal to } X, \\ m(X,X) &= \frac{n}{2}\mu. \end{aligned} \tag{4.18}$$

(iii) The equality case of (4.17) holds for all $X \in T_p^1(M)$ iff either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.

Proof. From (4.16), we get

$$\begin{aligned} \frac{1}{4}\varphi n^2\mu^2 &= r_{S(TM)}(p) - \frac{n(n-1)}{2}c + (n-1)\lambda + \frac{1}{4}\varphi(m_{11} - m_{22} - ... - m_{nn})^2 \\ &+\varphi\sum_{j=2}^{n}(m_{1j})^2 - \varphi\sum_{2\leq i<j\leq n} m_{ii}m_{jj} - (m_{ij})^2. \end{aligned} \tag{4.19}$$

Using (3.10) we obtain

$$\varphi\sum_{2\leq i<j\leq n} m_{ii}m_{jj} - (m_{ij})^2 = \sum_{2\leq i<j\leq n} K_{ij} - \frac{(n-2)(n-1)}{2}c + (n-2)\left(\lambda - \alpha(e_1,e_1)\right). \tag{4.20}$$

From (4.19) and (4.20), we have

$$\begin{aligned} Ric_{S(TM)}(e_1) &= \frac{1}{4}\varphi n^2\mu^2 + (n-1)c - (2n-3)\lambda - \frac{1}{4}\varphi(m_{11} - m_{22} - ... - m_{nn})^2 \\ &-\varphi\sum_{j=2}^{n}(m_{1j})^2 + (n-2)\alpha(e_1,e_1). \end{aligned} \tag{4.21}$$

If we take $e_1 = X$ like any vector of $T_p^1(M)$ in (4.21) one gets (4.17).
Equality holds in (4.17) for $X \in T_p^1(M)$ iff

$$m_{12} = m_{13} = ... = m_{1n} = 0 \text{ and } m_{11} = m_{22} + ... + m_{nn}, \tag{4.22}$$

which is quivalent to (4.18).

Now we prove the statement (iii). Assuming the equality in (4.17) for all $X \in T_p^1(M)$, in view of (4.22), we have

$$m_{ij} = 0, \quad i \neq j. \tag{4.23}$$

$$2m_{ii} = m_{11} + m_{22} + ... + m_{nn}, \quad i \in \{1, ..., n\}. \tag{4.24}$$

From (4.24), we have $2m_{11} = 2m_{22} = ... = 2m_{nn} = m_{11} + m_{22} + ... + m_{nn}$, which implies that

$$(n-2)(m_{11} + m_{22} + ... + m_{nn}) = 0.$$

Thus, either $m_{11} + m_{22} + ... + m_{nn} = 0$ or $n = 2$. If $m_{11} + m_{22} + ... + m_{nn} = 0$, then from (4.24), we get

$$m_{ii} = 0 \text{ for all } i \in \{1, ..., n\}.$$

By the above equation and (4.23), we obtain $m_{ij} = 0$ for all $i, j \in \{1, ..., n\}$, that imlies that p is a totally geodesic point. If $n = 2$, then from (4.24), $2m_{11} = 2m_{22} = m_{11} + m_{22}$, that is, p is a totally umbilical point. Converse is trivial. □

Lemma 4.2. *If $n > k \geq 2$ and $a_1, ..., a_n \in \mathbb{R}$ are real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-k+1)\left(\sum_{i=1}^{n} a_i^2 + a\right),$$

then

$$2\sum_{1 \leq i < j \leq k}^{n} a_i a_j \geq a.$$

with equality holding iff

$$a_1 + a_2 + ... + a_k = a_{k+1} = ... = a_n.$$

Theorem 4.4. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ such that the vector field P is tangent to M. Then, for each point $p \in M$ and each non-degenerate k-plane section $\Pi_k \subset TpM$ $(n > k \geq 2)$, we have*

$$\begin{aligned} \tau_{S(TM)}(p) - \tau(\pi_k) \quad &\geq \quad (n-k)\left(\frac{\varphi n^2}{(n-k+1)}\mu^2 + (n-k+1)c - \lambda\right) \\ &\quad -\varphi\sum_{r=k}^{n}(m_{ii})^2 + 2(k-1)trace(\alpha|_{\pi_k^{\perp}}). \end{aligned} \tag{4.25}$$

If the equality case of (4.25) satisfies for $p \in M$, thus M is minimal and the form of shape operator of M becomes

$$A_\xi^* = \begin{bmatrix} m_{11} & m_{12} & . & . & m_{1k} & \\ m_{21} & m_{22} & . & . & m_{2k} & \\ . & . & . & . & . & 0 \\ . & . & . & . & . & \\ . & . & . & . & . & \\ m_{k1} & m_{k2} & . & . & -\sum_{i=1}^{k-1}(m_{ii}) & \\ & & 0 & & & 0_{n-k} \end{bmatrix}. \tag{4.26}$$

Proof. One takes

$$\varepsilon = \tau_{S(TM)}(p) - n(n-1)c + 2(n-1)\lambda - \varphi\frac{n^2(n-k)}{(n-k+1)}\mu^2, \tag{4.27}$$

in (4.8), then we have

$$\varepsilon = \varphi\frac{n^2}{(n-k+1)}\mu^2 - \varphi\sum_{i,j=1}^{n}(m_{ij})^2.$$

Therefore, we can write

$$\left(\sum_{i=1}^{n} m_{ii}\right)^2 = (n-k+1)\left(\sum_{i=1}^{n}(m_{ii})^2 + \sum_{i\neq j=1}^{n}(m_{ij})^2 + \frac{\varepsilon}{\varphi}\right). \tag{4.28}$$

From Lemma 4.2 we get

$$2\sum_{1\leq i<j\leq k} m_{ii}m_{jj} \geq \sum_{i\neq j=1}^{n}(m_{ij})^2 + \frac{\varepsilon}{\varphi}. \tag{4.29}$$

Now, a non-degenerate plane section π_k spanned by $\{e_1, e_2, ..., e_k)$. Then one obtains

$$\begin{aligned}
\tau(\pi_k) &= k(k-1)c - \sum_{i,j=1}^{k}(\alpha(e_i,e_i)+\alpha(e_j,e_j)) + \varphi\sum_{i,j=1}^{k} m_{ii}m_{jj} - (m_{ij})^2 \\
&= k(k-1)c - \sum_{i,j=1}^{k}(\alpha(e_i,e_i)+\alpha(e_j,e_j)) + \varphi\sum_{i=1}^{k}(m_{ii})^2 \\
&\quad +2\varphi\sum_{1\leq i<j\leq k} m_{ii}m_{jj} - \varphi\sum_{i,j=1}^{k}(m_{ij})^2 \\
&\geq k(k-1)c - 2(k-1)\sum_{i=1}^{k}\alpha(e_i,e_i) + \varepsilon + \sum_{i\neq j=1}^{n}(m_{ij})^2 - \varphi\sum_{i,j=1}^{k}(m_{ij})^2 \\
&\geq k(k-1)c - 2(k-1)\sum_{i=1}^{k}\alpha(e_i,e_i) + \varepsilon + \varphi\sum_{i,j=1}^{n}(m_{ij})^2 - \varphi\sum_{i=1}^{n}(m_{ii})^2 - \varphi\sum_{i,j=1}^{k}(m_{ij})^2 \\
&\geq k(k-1)c - 2(k-1)\sum_{i=1}^{k}\alpha(e_i,e_i) + \varepsilon + \varphi\sum_{i,j=k+1}^{n}(m_{ij})^2 - \varphi\sum_{i=k}^{n}(m_{ii})^2.
\end{aligned} \tag{4.30}$$

We remark that

$$\sum_{i=1}^{k}\alpha(e_i,e_i) = \lambda - trace(\alpha|_{\pi_k^{\perp}}). \tag{4.31}$$

Using (4.27), (4.30) and (4.31) we get

$$\begin{aligned}
\tau(\pi_k) &\geq k(k-1)c - 2(k-1)\left(\lambda - trace(\alpha|_{\pi_k^{\perp}})\right) - \varphi\sum_{i=k}^{n}(m_{ii})^2 \\
&\quad +\tau_{S(TM)}(p) - n(n-1)c + 2(n-1)\lambda - \varphi\frac{n^2(n-k)}{n-k+1}\mu^2.
\end{aligned} \tag{4.32}$$

From (4.32) we have (4.25) and (4.26) which implies that M is minimal. □

Furthermore, the second fundamental form m and the screen second fundamental form D provide

$$\sum_{i,j=1}^{n} m_{ij}D_{ji} = \frac{1}{2}\left\{\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2 - \sum_{i,j=1}^{n}(m_{ij})^2 + (D_{ji})^2\right\} \tag{4.33}$$

and

$$\sum_{i,j=1}^{n} m_{ii}D_{jj} = \frac{1}{2}\left\{\left(\sum_{i,j=1}^{n} m_{ii}+D_{jj}\right)^2 - \left(\sum_{i=1}^{n} m_{ii}\right)^2 - \left(\sum_{j=1}^{n} D_{jj}\right)^2\right\}. \tag{4.34}$$

Theorem 4.5. *Let M be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, we have*

(i)

$$\tau_{S(TM)}(p) \leq n(n-1)c - 2(n-1)\lambda + n\mu trace A_N + \frac{1}{2}\sum_{i,j=1}^{n}(m_{ij})^2 + (D_{ji})^2. \tag{4.35}$$

The equality case of (4.35) satisfies for all $p \in M$ iff either M is a screen homothetic lightlike hypersurface with $\varphi = -1$ or M is a totally geodesic lightlike hypersurface.

(ii)

$$\tau_{S(TM)}(p) \geq n(n-1)c - 2(n-1)\lambda + n\mu trace A_N - \frac{1}{2}\sum_{i,j=1}^{n}(m_{ij})^2 + (D_{ji})^2. \tag{4.36}$$

The equality case of (4.36) satisfies for all $p \in M$ iff either M is a screen homothetic lightlike hypersurface with $\varphi = 1$ or M is a totally geodesic lightlike hypersurface.

(iii) (4.35) and (4.36) with equalities iff p is a totally geodesic point.

Proof. From (4.7) and (4.33), we get

$$\tau_{S(TM)}(p) = n(n-1)c - 2(n-1)\lambda + \sum_{i,j=1}^{n} m_{ii}D_{jj} - \frac{1}{2}\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2 + \frac{1}{2}\sum_{i,j=1}^{n}(m_{ij})^2 + (D_{ji})^2 \tag{4.37}$$

which yields (4.35).

Since

$$\frac{1}{2}((m_{ij})^2 + (D_{ji})^2) = \frac{1}{4}(m_{ij}+D_{ji})^2 + \frac{1}{4}(m_{ij}-D_{ji})^2, \tag{4.38}$$

one obtains

$$\tau_{S(TM)}(p) = n(n-1)c - 2(n-1)\lambda + \sum_{i,j=1}^{n} m_{ii}D_{jj} + \frac{1}{2}\sum_{i,j=1}^{n}(m_{ij}-D_{ji})^2 - \frac{1}{2}\sum_{i,j=1}^{n}(m_{ij})^2 + (D_{ji})^2 \tag{4.39}$$

which implies (4.36). From (4.35), (4.36), (4.37) and (4.39) (i), (ii) and (iii) statements are easily obtained. □

Thus we get the following corollary.

Corollary 4.1. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, we have*

(i)

$$\tau_{S(TM)}(p) \leq n(n-1)c - 2(n-1)\lambda + \varphi n^2\mu^2 + \left(\frac{1+\varphi^2}{2}\right)\sum_{i,j=1}^{n}(m_{ij})^2. \tag{4.40}$$

(ii)

$$\tau_{S(TM)}(p) \geq n(n-1)c - 2(n-1)\lambda + \varphi n^2\mu^2 - \left(\frac{1+\varphi^2}{2}\right)\sum_{i,j=1}^{n}(m_{ij})^2. \tag{4.41}$$

Theorem 4.6. *Let M be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, we derive*

$$\begin{aligned}\tau_{S(TM)}(p) \quad \leq \quad & n(n-1)c - 2(n-1)\lambda + \frac{1}{2}(trace\bar{A})^2 - \frac{1}{2}(trace A_N)^2 \\ & -\frac{1}{4}\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2 + \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij}-D_{ji})^2,\end{aligned} \tag{4.42}$$

where

$$\bar{A} = \begin{bmatrix} m_{11}+D_{11} & m_{12}+D_{21} & . & . & . & m_{1n}+D_{n1} \\ m_{21}+D_{12} & m_{22}+D_{22} & & . & . & m_{2n}+D_{n2} \\ . & & . & & & \\ . & & & & & \\ . & & & & & \\ m_{n1}+D_{1n} & m_{n2}+D_{2n} & & & & m_{nn}+D_{nn} \end{bmatrix}. \tag{4.43}$$

For all $p \in M$ the equality case of (4.42) *satisfies iff M is minimal.*

Proof. From (4.7), (4.33) and (4.34) we obtain

$$\begin{aligned} \tau_{S(TM)}(p) &= n(n-1)c - 2(n-1)\lambda + \frac{1}{2}\sum_{i,j=1}^{n}(m_{ii}+D_{jj})^2 - \frac{1}{2}\left(\sum_{i=1}^{n} m_{ii}\right)^2 \\ &\quad -\frac{1}{2}\left(\sum_{j=1}^{n} D_{jj}\right)^2 - \frac{1}{2}\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2 + \frac{1}{2}\sum_{i,j=1}^{n}(m_{ij})^2 + (D_{ji})^2. \end{aligned} \tag{4.44}$$

From (4.38) we have

$$-\frac{1}{2}\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2 + \frac{1}{2}\sum_{i,j=1}^{n}(m_{ij})^2 + (D_{ji})^2 = -\frac{1}{4}\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2 + \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij}-D_{ji})^2. \tag{4.45}$$

Using (4.45) in (4.44), we get

$$\begin{aligned} \tau_{S(TM)}(p) &= n(n-1)c - 2(n-1)\lambda + \frac{1}{2}\sum_{i,j=1}^{n}(m_{ii}+D_{jj})^2 - \frac{1}{2}\left(\sum_{i=1}^{n} m_{ii}\right)^2 \\ &\quad -\frac{1}{2}\left(\sum_{j=1}^{n} D_{jj}\right)^2 - \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2 + \frac{1}{4}\sum_{i,j=1}^{n}(m_{ij}-D_{ji})^2. \end{aligned} \tag{4.46}$$

Assume the equality case of (4.42) is satisfied, then

$$\sum_{i} m_{ii} = 0.$$

Thus M is minimal. ☐

Thus we get the following corollary.

Corollary 4.2. *Let M be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, we get*

$$\tau_{S(TM)}(p) \le n(n-1)c - 2(n-1)\lambda + \left(\frac{2\varphi+1}{2}\right)n^2\mu^2 - \varphi\sum_{i=1}^{n}(m_{ij})^2. \tag{4.47}$$

For all $p \in M$ the equality case of (4.47) satisfies iff M is minimal.

Theorem 4.7. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, we derive*

$$\begin{aligned} \tau_{S(TM)}(p) &\le n(n-1)c - 2(n-1)\lambda + \frac{(n-1)}{2n}(trace\bar{A})^2 - \frac{1}{2}(traceA_N)^2 \\ &\quad -\frac{1}{2}n^2\mu^2 - \frac{1}{2}\sum_{i\neq j}(m_{ij}+D_{ji})^2 + \frac{1}{2}\sum_{i,j=1}^{n}(m_{ij})^2 + (D_{ji})^2, \end{aligned} \tag{4.48}$$

where $\bar{A}$ is equal to (4.43).

For all $p \in M$ the equality case of (4.48) *satisfies iff $n\mu = -traceA_N$.*

Proof. From (4.44),we get

$$\begin{aligned}\tau_{S(TM)}(p) &= n(n-1)c-2(n-1)\lambda+\frac{1}{2}(trace\bar{A})^2-\frac{1}{2}(traceA_N)^2-\frac{1}{2}n^2\mu^2\\ &-\frac{1}{2}\sum_{i=1}^{n}(m_{ii}+D_{ii})^2-\frac{1}{2}\sum_{i\neq j}(m_{ij}+D_{ji})^2+\frac{1}{2}\sum_{i,j=1}^{n}(m_{ij})^2+(D_{ji})^2. \end{aligned} \tag{4.49}$$

Using Lemma 4.1 and equality case of (4.49), we have

$$\begin{aligned}\tau_{S(TM)}(p) &\leq n(n-1)c-2(n-1)\lambda+\frac{1}{2}(trace\bar{A})^2-\frac{1}{2}(traceA_N)^2-\frac{1}{2}n^2\mu^2\\ &-\frac{1}{2n}\left(\sum_{i=1}^{n}m_{ii}+D_{ii}\right)^2-\frac{1}{2}\sum_{i\neq j}(m_{ij}+D_{ji})^2+\frac{1}{2}\sum_{i,j=1}^{n}(m_{ij})^2+(D_{ji})^2 \end{aligned} \tag{4.50}$$

which implies (4.48). The equality case of (4.48) holds, then

$$m_{11}+D_{11}=m_{22}+D_{22}=...=m_{nn}+D_{nn}. \tag{4.51}$$

From (4.51) we obtain

$$\begin{aligned}(1-n)m_{11}+m_{22}+...+m_{nn}+(1-n)D_{11}+D_{22}+...+D_{nn} &= 0,\\ m_{11}+(1-n)m_{22}+...+m_{nn}+D_{11}+(1-n)D_{22}+...+D_{nn} &= 0,\\ &.\\ &.\\ &.\\ m_{11}+m_{22}+...+(1-n)m_{nn}+D_{11}+D_{22}+...+(1-n)D_{nn} &= 0.\end{aligned}$$

Using last equations, we have

$$(n-1)^2(traceA_N+n\mu)=0. \tag{4.52}$$

Because of $n\neq 1$,we get $n\mu=-traceA_N$. □

Thus we get the following corollary.

Corollary 4.3. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, we have*

$$\tau_{S(TM)}(p)\leq n(n-1)c-2(n-1)\lambda+\varphi n(n-1)\mu^2-\frac{(1+\varphi^2)}{2}n\mu^2-\varphi\sum_{i\neq j}^{n}(m_{ij})^2+\frac{(1+\varphi^2)}{2}\sum_{i=1}^{n}(m_{ii})^2. \tag{4.53}$$

For all $p\in M$ the equality case of (4.53) satisfies iff either $\varphi=-1$ or M is minimal.

Theorem 4.8. *Let M be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then*

$$\begin{aligned}\tau_{S(TM)}(p) &\geq n(n-1)c-2(n-1)\lambda+\frac{1}{2}\left((trace\bar{A})^2-(traceA_N)^2-n(n-1)\mu^2\right)\\ &+\frac{1}{2}\left(\sum_{i\neq j=1}^{n}(m_{ij})^2-\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2+\sum_{i,j=1}^{n}(D_{ji})^2\right). \end{aligned} \tag{4.54}$$

For all $p\in M$ the equality case of (4.54) satisfies iff p is a totally umbilical point.

Proof. Using (4.44),we get

$$\begin{aligned}\tau_{S(TM)}(p) &= n(n-1)c-2(n-1)\lambda+\frac{1}{2}(trace\bar{A})^2-\frac{1}{2}(traceA_N)^2-\frac{1}{2}n^2\mu^2\\ &+\frac{1}{2}\sum_{i=1}^{n}(m_{ii})^2+\frac{1}{2}\sum_{i\neq j=1}^{n}(m_{ij})^2+\frac{1}{2}\sum_{i,j=1}^{n}(D_{ji})^2-\frac{1}{2}\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2. \end{aligned} \tag{4.55}$$

Using Lemma 4.2 and equality case of (4.44), we have

$$\begin{aligned}\tau_{S(TM)}(p) \geq\ & n(n-1)c - 2(n-1)\lambda + \frac{1}{2}(trace\bar{A})^2 - \frac{1}{2}(traceA_N)^2 - \frac{1}{2}n^2\mu^2 \\ & -\frac{1}{2}\sum_{i,j=1}^{n}(m_{ij}+D_{ji})^2 + \frac{1}{2n}(\sum_{i=1}^{n}m_{ii})^2 + \frac{1}{2}\sum_{i\neq j=1}^{n}(m_{ij})^2 + \frac{1}{2}\sum_{i,j=1}^{n}(D_{ji})^2 \end{aligned} \tag{4.56}$$

which implies (4.53). The equality case of (4.43) satisfies iff

$$m_{11} = ... = m_{nn}$$

and the shape operator A^*_ξ becomes of the form

$$A^*_\xi = \begin{bmatrix} m_{11} & 0 & . & . & . & 0 & 0 \\ 0 & m_{11} & . & . & . & 0 & 0 \\ . & & . & & & & \\ . & & & . & & & \\ . & & & & . & & \\ 0 & 0 & . & . & . & m_{11} & 0 \\ 0 & 0 & . & . & . & 0 & 0 \end{bmatrix}, \tag{4.57}$$

which indicates that M is totally umbilical. Hence, the claim holds. □

Thus we get the following corollary.

Corollary 4.4. *Let M be an $(n+1)$-dimensional screen homothetic lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, we get*

$$\tau_{S(TM)}(p) \geq n(n-1)c - 2(n-1)\lambda + \frac{(2\varphi+1)}{2}n^2\mu^2 - \frac{n(n-1)}{2}\mu^2 - \frac{(2\varphi+1)}{2}\sum_{i,j=1}^{n}(m_{ij})^2. \tag{4.58}$$

For all $p \in M$ the equality case of (4.58) satisfies iff p is a totally umbilical point.

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