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Construction of the Null Scrolls along Lightlike Submanifolds in $\mathbb{R}_v^{m+n}$

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ABSTRACT

The theory of null scrolls is still a developing subject. In this paper, the aim is to generalize the null scrolls by developing a method. For this, it is constructed the null scrolls along lightlike submanifolds in $\mathbb{R}_v^{m+n}$. Several geometric objects of the defined null scrolls are investigated. The proposed theory is strengthened with examples.

Keywords: *Lightlike submanifold; null scroll; second fundamental form.*

AMS Subject Classification (2010): *Primary: 53C50; 58A05 ; Secondary: 53A35 ; 53C05.*

1. Introduction

It is known that, a ruled surface is generated by a base curve and a non-zero director vector field. Also, the ruled surfaces are classified in five different kinds according to the casual characters of the base curve and the director vector field in a semi- Riemannian manifold. If both of them are null, then the ruled surface is called a null scroll [8, 10].

In the semi-Riemannian geometry, the theory of lightlike submanifolds plays an important role. It takes a special place not only in Geometry but also in Physics. In [1], the authors extended the theory of null scrolls by defining n-dimensional generalized null scrolls in $\mathbb{R}_1^n$. However, in the present construction method, it is considered the lightlike submanifolds in order to extend the theory to the higher dimensions and arbitrary indexes. For this, a lightlike normal vector field in the transversal vector bundle is defined by using the bases of $RadTM$, $ltr(TM)$ and $S(TM^\perp)$. It is described by the sum of a timelike unit normal vector field and a spacelike unit normal vector field from the unit spherical bundle. Then the null scroll is defined in two cases depending on the casual characters of the vector fields in the basis of $S(TM^\perp)$ by,

$$Y_M(u,\theta,t) = X(u) + t(n^T + n^S)(u),\, t \in \mathbb{R}$$

where $X(U) = M$ is the lightlike submanifold, $u = (u_1, ..., u_m)$, $\theta = (\theta_1, ..., \theta_{n-2})$. Obviously, a more general description of the null scrolls is given in this study. Moreover, the conditions for being flat and the relation between the second fundamental forms of the lightlike submanifold and corresponding null scroll, are given. The theory is strengthened with some examples.

2. Preliminaries

In this section, the basic properties of the theory of lightlike submanifolds are mentioned (see [3,4]). Let $\bar{M}$ be a $m + n$ dimensional semi-Riemannian manifold and $\bar{g}$ be the metric with constant index defined on $\bar{M}$. When M is a m dimensional submanifold in $\bar{M}$, for every $p \in M$, consider a subspace

$$T_pM^\perp = \left\{v_p \in T_p\bar{M} : \bar{g}_p\left(v_p, w_p\right) = 0,\ \forall w_p \in T_pM\right\}.$$

Then the radical distribution is defined by,

$$RadT_pM = T_pM \cap T_pM^\perp \neq \{0\},\ \forall p \in M.$$

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If the rank of $RadTM$ is $r > 0$, then M is called a *r-lightlike submanifold*.

The complement vector bundle to $RadTM$ in TM is $S(TM)$ which is called a *screen distribution*. Clearly, $S(TM)$, is a non-degenerate for $\bar{g}$. Hence, one can write the following decomposition,

$$TM = RadTM \oplus_{ort} S(TM).$$

$RadTM = TM \cap TM^{\perp}$ where $TM^{\perp} = \underset{p \in M}{\cup} T_p M^{\perp}$.

Theorem 2.1. *Let* $\left(M, g, S(TM), S(TM^{\perp})\right)$ *be a r-lightlike submanifold in* $\bar{M}$. *If* U *is a coordinate neighborhood of* M *and* $\{\xi_i\}$ $(i = 1, ..., r)$ *is a basis of* $RadTM$. *Then, there exist smooth sections* $\{N_i\}$ *where the following equations hold:*

$$\bar{g}\left(\xi_i, N_i\right) = \delta_{ij}$$

$$\bar{g}\left(N_i, N_j\right) = 0, \ \ i, j = 1, ..., r$$

$ltr(TM) = Span\{N_i\}$ is called *lightlike transversal vector bundle*.

The following decomposition is also satisfied;

$$T\bar{M}\mid_M = TM \oplus tr(TM)$$

where $tr(TM) = ltr(TM) \oplus S(TM^{\perp})$. In this case, a quasi-ortonormal basis of $\bar{M}$ along M is

$$\{\xi_1, ..., \xi_r, N_1, ..., N_r, X_{r+1}, ..., X_m, W_{r+1}, ..., W_n\}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $RadTM$, $\{N_1, ..., N_r\}$ is a basis of $ltr(TM)$, $\{X_{r+1}, ..., X_m\}$ and $\{W_{r+1}, ..., W_n\}$ are the orthogonal basis of $S(TM)$ and $S(TM^{\perp})$ respectively.

Then, it can be written

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V$$

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$, $\nabla_X Y$ and $\nabla^t_X V$ are the linear connections on M and $tr(TM)$ respectively. Note that ∇ is a torsion free induced linear connection. Also, $A_V X$ and $h(X, Y)$ are the shape operator and second fundamental form on M, respectively.

L and S are the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively where $S(TM^{\perp}) \neq \{0\}$. In this case, h^l and h^s are the lightlike and screen second fundamental forms respectively defined by

$$h^l(X, Y) = L(h(X, Y)) \text{ and } h^s(X, Y) = S(h(X, Y))$$

$$h(X, Y) = h^l(X, Y) + h^s(X, Y).$$

3. Null Scrolls Along Lightlike Submanifolds

Let $X(U) = M$ be a m dimensional $r - lightlike$ submanifold in $\mathbb{R}^{m+n}_v$ where $n \geq 2$ and $v \leq m$. $\{\xi_1, ..., \xi_r\}$ is taken as the basis of $Rad(TM)$. Then, it is considered that $S(TM^{\perp})$ is spanned by the vector fields $\{W_{r+1}, ..., W_n\}$. The pseudo orthonormal basis of $S(TM)^{\perp}$ along M is,

$$\{\xi_1, ..., \xi_r, N_1, ..., N_r, W_{r+1}, ...W_n\}$$

where $\{N_1, ..., N_r\}$ is a basis of $ltr(TM)$. Since $S(TM^{\perp})$ is a non-degenerate subspace, two cases arise:

Case 1 Let all the vector fields $\{W_{r+1}, ..., W_n\}$ be spacelike (timelike). Then, without loss of generality, choose a unit timelike (spacelike) vector field $n^T(u) = \dfrac{a_1\xi_1(u) + b_1 N_1(u)}{\|a_1\xi_1(u) + b_1 N_1(u)\|}$ where $a_1, b_1 \in \mathbb{R}$. On the other hand, take following orthogonal unit spacelike (timelike) vector fields

$$U_{k-1}(u) = \frac{a_k\xi_k(u) + b_k N_k(u)}{\|a_k\xi_k(u) + b_k N_k(u)\|}$$

where $a_k, b_k \in \mathbb{R}$ and $k = 2, ..., r$. Then, one gets a non degenerate subspace $W_p = Span\{U_1(p), ..., U_{r-1}(p), W_{r+1}(p), ..., W_n(p)\}$ in $S(TM)_p^{\perp}$ at $p = X(u)$. Therefore, a spacelike (timelike) unit pseudo sphere is,

$$S_p^{n-1} = \{v \in W_p : \langle v, v\rangle = 1\}.$$

Hence, there exists a pseudo spherical bundle $S^{n-1} = \underset{p\in M}{\cup} S_p^{n-1}$ over M. A unit spacelike vector field $n^S(u) \in S^{n-1}$ can be written by means of pseudo spherical parameters θ_j where $j = 1, ..., n-2$. It is clear that $\langle n^S, n^T\rangle = 0$. The vector fields $n^T(u) \pm n^S(u)$ are lightlike. $n^T + n^S$ is taken as a lightlike transversal normal vector field along M.

Definition 3.1. The hypersurface

$$Y_M : S^{n-1} \times \mathbb{R} \longrightarrow \mathbb{R}_v^{m+n}$$

defined by

$$Y_M(u, \theta, t) = X(u) + t(n^T + n^S)(u), t \in \mathbb{R}$$

is called a null scroll along M where $u = (u_1, ..., u_m)$, $\theta = (\theta_1, ..., \theta_{n-2})$.

Case 2 Now let $\{W_{r+1}, ...W_s\}$ be spacelike and $\{W_{s+1}, ..., W_n\}$ be timelike vector fields. Then, without loss of generality, choose some orthogonal timelike vector fields as;

$$n_1(u) = \frac{a_1\xi_1(u) + b_1N_1(u)}{\|a_1\xi_1(u) + b_1N_1(u)\|}$$

$$n_2(u) = \frac{a_2\xi_2(u) + b_2N_2(u)}{\|a_2\xi_2(u) + b_2N_2(u)\|}$$

$$\vdots$$

$$n_\mu(u) = \frac{a_\mu\xi_\mu(u) + b_\mu N_\mu(u)}{\|a_\mu\xi_\mu(u) + b_\mu N_\mu(u)\|}$$

where $a_1, ..., a_\mu, b_1, ..., b_\mu \in \mathbb{U}211d$, $\mu < r$. Similarly, take the following orthogonal spacelike vector fields;

$$U_1(u) = \frac{a_{\mu+1}\xi_{\mu+1}(u) + b_{\mu+1}N_{\mu+1}(u)}{\|a_{\mu+1}\xi_{\mu+1}(u) + b_{\mu+1}N_{\mu+1}(u)\|}$$

$$U_2(u) = \frac{a_{\mu+2}\xi_{\mu+2}(u) + b_{\mu+2}N_{\mu+2}(u)}{\|a_{\mu+2}\xi_{\mu+2}(u) + b_{\mu+2}N_{\mu+2}(u)\|}$$

$$\vdots$$

$$U_{r-\mu}(u) = \frac{a_r\xi_r(u) + b_rN_r(u)}{\|a_r\xi_r(u) + b_rN_r(u)\|}$$

where $a_{\mu+1}, ..., a_r, b_{\mu+1}, ..., b_r \in \mathbb{R}$. Then, one gets non degenerate subspaces as follows;

$$W_p^1 = Span\{n_1(p), ..., n_\mu(p), W_{s+1}(p), ..., W_n(p)\}$$

$$W_p^2 = Span\{U_1(p), ..., U_{r-\mu}(p), W_{r+1}(p), ..., W_s(p)\}.$$

Also, the unit pseudo spheres S_p^ν and H_p^ϑ in W_p^2 and W_p^1 respectively, are defined by

$$S_p^\varsigma = \{v \in W_p^2 : \langle v, v\rangle = 1\}$$

$$H_p^\vartheta = \{v \in W_p^1 : \langle v, v\rangle = -1\}$$

where $\varsigma = s - \mu - 1$, $\vartheta = n - s + \mu - 1$. Hence, there exist pseudo spherical bundles $S^\varsigma = \underset{p\in M}{\cup} S_p^\varsigma$ and $H^\vartheta = \underset{p\in M}{\cup} H_p^\vartheta$ over M. A unit spacelike vector field $n^S(u) \in S^\varsigma$ and a unit timelike vector field $n^T(u) \in H^\vartheta$ can be written by means of pseudo spherical parameters ψ_j and φ_l, respectively where $j = 1, ..., \varsigma - 1$, $l = 1, ..., \vartheta - 1$. It is clear that, $\langle n^S, n^T\rangle = 0$. The vector fields $n^T(u) \pm n^S(u)$ are lightlike. $n^T + n^S$ is taken as a lightlike transversal normal vector field along M.

Definition 3.2. The hypersurface

$$Y_M : S^{n-1} \times \mathbb{R} \longrightarrow \mathbb{R}_v^{m+n}$$

defined by

$$Y_M(u, \psi, \varphi, t) = X(u) + t(n^T + n^S)(u), t \in \mathbb{R}$$

is called a null scroll along M where $u = (u_1, ..., u_m), \varphi = (\varphi_1, ..., \varphi_{\vartheta-1}), \psi = (\psi_1, ..., \psi_{\varsigma-1})$ and $\vartheta + \varsigma = n$.

Whether or not it belongs to the Case 1 or Case 2, the null scroll along M is denoted by $\tilde{M}$. The tangent space $T_P\tilde{M}$ at a point $P \in \tilde{M}$ is spanned by the following vector fields;

$$Y_{u_i}(u, \theta, t) = X_{u_i}(u) + t\left(n^T + n^S\right)_{u_i}(u)\,,\ i = 1, ..., m$$

$$Y_{\theta_j}(u, \theta, t) = t\left(n^S_{\theta_j}\right)(u)\,,\ j = 1, ..., n-2$$

$$Y_t(u, \theta, t) = \left(n^T + n^S\right)(u)\,.$$

Note that, it is taken $\theta = (\theta_1, ..., \theta_{n-2}) = (\varphi_1, ..., \varphi_{\vartheta-1}, \psi_1, ..., \psi_{\varsigma-1})$ if $\tilde{M}$ belongs to the Case 2. Therefore, Y_t is null and Y_{θ_j} are non-null vector fields. Y_{u_i} are spacelike, timelike or null vector fields.

Let Y_{u_α} be non-null vector fields and Y_{u_β} be null vector fields where $\alpha = 1, ..., q$ and $\beta = q+1, ..., m$. Take $a_1 = u_1, ..., a_m = u_m, a_{m+1} = \theta_1, ..., a_{m+n-2} = \theta_{n-2}, a_{m+n-1} = t$ for the easement of the calculations. The induced pseudo Riemannian metric which is the first fundamental form on $\tilde{M}$, is defined by $ds^2 = \sum\limits_{i,j=1}^{m+n-1} \tilde{g}_{ij} da_i da_j$ where $\tilde{g}_{ij} = \langle Y_{a_i}, Y_{a_j}\rangle$. The matrix form of $\tilde{g}_{ij}$ is,

$$\begin{bmatrix}
\varepsilon_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \varepsilon_2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \varepsilon_q & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \lambda_{q+1}^{q+2} & \cdots & \lambda_{q+1}^{m} & 0 & \cdots & 0 & 0 & \lambda_{m+n-1}^{q+1} \\
0 & 0 & \cdots & 0 & \lambda_{q+2}^{q+1} & 0 & \cdots & \lambda_{q+2}^{m} & 0 & \cdots & 0 & 0 & \lambda_{m+n-1}^{q+2} \\
\vdots & \vdots & & \vdots & \vdots & & \ddots & & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & & 0 & \lambda_{m}^{q+1} & \lambda_{m}^{q+2} & \cdots & 0 & 0 & \cdots & 0 & 0 & \lambda_{m+n-1}^{m} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \varepsilon_{m+1} & \cdots & 0 & 0 & 0 \\
0 & 0 & & 0 & 0 & 0 & & 0 & 0 & \varepsilon_{m+2} & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \varepsilon_{m+n-2} & 0 \\
0 & 0 & & 0 & \lambda_{m+n-1}^{q+1} & \lambda_{m+n-1}^{q+2} & \cdots & \lambda_{m+n-1}^{m} & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}$$

where $\varepsilon_k = \pm 1$ and $\lambda_i^j = 0$ or 1 depending on the null vector fields.

Now consider a normal vector field $V \in N_P\tilde{M}$ along $\tilde{M}$. Then the second fundamental form of $\tilde{M}$ is,

$$\tilde{h}_{ij} = \langle -\tilde{\nabla}_{Y_{a_i}} V, Y_{a_j}\rangle, i, j = 1, ..., m+n-1$$

where $\tilde{\nabla}$ is the induced connection on $\tilde{M}$.

Theorem 3.1. *A null scroll along a r-lightlike submanifold is flat.*

Theorem 3.2. *A null scroll along a r -lightlike submanifold is Ricci-flat.*

Proposition 3.1. *Let $\tilde{M}$ be a null scroll along a $r-lightlike$ submanifold M. Then, the Weingarten formulas on $\tilde{M}$ are,*

$$\tilde{\nabla}_{Y_{a_i}} V = -\sum_{j=1}^{m+n-1} \tilde{h}_i^j Y_{a_j}, i = 1, ..., m+n-1$$

where $\tilde{h}_i^j = \tilde{h}_{ik}\tilde{g}^{kj}$ and $\tilde{g}^{kj} = (\tilde{g}_{kj})^{-1}$.

Theorem 3.3. *For a null scroll along a $r-lightlike$ submanifold, the following equations hold;*

$$\begin{aligned} \tilde{h}^{\sigma}_{\alpha} &= \tilde{h}^{\alpha}_{\alpha}\tilde{h}^{\sigma}_{\sigma}\varepsilon_{\alpha}\varepsilon_{\sigma} \\ \tilde{h}^{j}_{i} &= \tilde{h}^{i}_{i}\tilde{h}^{j}_{j}\varepsilon_{i}\varepsilon_{j} \\ \tilde{h}^{j}_{j}\tilde{h}^{\alpha}_{\alpha} &= 0 \end{aligned}$$

where $i,j = m+1,...,m+n-2$, $\alpha,\sigma = 1,...,q$.

Theorem 3.4. *Let h^{l}_{ij} and h^{s}_{ij} be the lightlike and screen second fundamental forms of a $r-lightlike$ submanifold M respectively. Then, they are related to the second fundamental form of the corresponding null scroll by following equation:*

$$\begin{aligned} \tilde{h}_{ij} &= \sum_{l=1}^{r} c_l \left[h^{l}_{ij} + t\sum_{k=1}^{m-r}\left(\eta_{jk}h^{l}_{ik} + \eta_{ik}h^{l}_{kj} + t\eta_{ik}\eta_{jk}h^{l}_{kk}\right)\right] \\ &\quad + \sum_{s=r+1}^{n} d_s \left[h^{s}_{ij} + t\sum_{k=1}^{m-r}\left(\eta_{jk}h^{s}_{ik} + \eta_{ik}h^{s}_{kj} + t\eta_{ik}\eta_{jk}h^{s}_{kk}\right)\right] \end{aligned}$$

where $i,j = 1,...,m$ *and* $c_l, d_s, \eta_{ik} \in \mathbb{R}$.

Corollary 3.1. *If all the null vector fields in the pseudo-frame of the tangent space at every point on the null scroll $\tilde{M}$ are orthogonal, then $\tilde{M}$ is totally umbilical.*

Theorem 3.5. *A null scroll along a r-lightlike submanifold is totally geodesic iff;*

$$\tilde{h}^{\alpha}_{i} = 0$$

$$\tilde{h}^{j}_{i} = 0$$

$$\sum_{k=q+1}^{m} \tilde{h}^{k}_{i}\lambda^{k}_{\beta} = 0$$

$$\sum_{k=q+1}^{m} \tilde{h}^{k}_{i}\lambda^{k}_{(m+n-1)} = 0$$

where $i = 1,...,m+n-1$, $j = m+1,...,m+n-1$, $\alpha = 1,...,q$, $\beta = q+1,...,m$.

Corollary 3.2. *Let M be a totally geodesic $r-lightlike$ submanifold. Then, the vector fields Y_{u_i} $(i = 1,...,m)$ are asymptotic directions of the corresponding null scroll $\tilde{M}$.*

4. Examples

Example 4.1. Take the following null curve as the lightlike submanifold M,

$$\alpha(s) = \left(\frac{1}{2}\sinh(2s), \frac{1}{2}\cosh(2s), s\right).$$

The Cartan frame along α is,

$$L = (\cosh(2s), \sinh(2s), 1)$$

$$N = \left(\frac{1}{2}\cosh(2s), \frac{1}{2}\sinh(2s), -\frac{1}{2}\right)$$

$$W = (\sinh(2s), \cosh(2s), 0).$$

Choose $n^T = \frac{L+N}{\|L+N\|} = \left(\frac{3}{2\sqrt{2}}\cosh(2s), \frac{3}{2\sqrt{2}}\sinh(2s), \frac{1}{2\sqrt{2}}\right)$ and $n^S = W$. Therefore, the vector field

$$n^T + W = \left(\frac{3}{2\sqrt{2}}\cosh(2s) + \sinh(2s), \frac{3}{2\sqrt{2}}\sinh(2s) + \cosh(2s), \frac{1}{2\sqrt{2}}\right)$$

is a lightlike transversal normal vector field along $\alpha(s)$. Hence, the null scroll along $\alpha(s)$ is;

$$Y(s,t) = \alpha(s) + t\left(n^T + W\right).$$

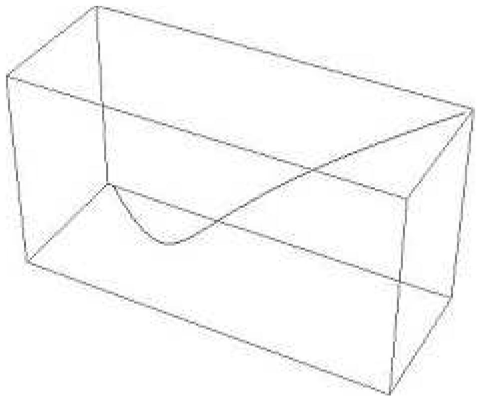

Figure 1. Curve $\alpha\left(s\right)$

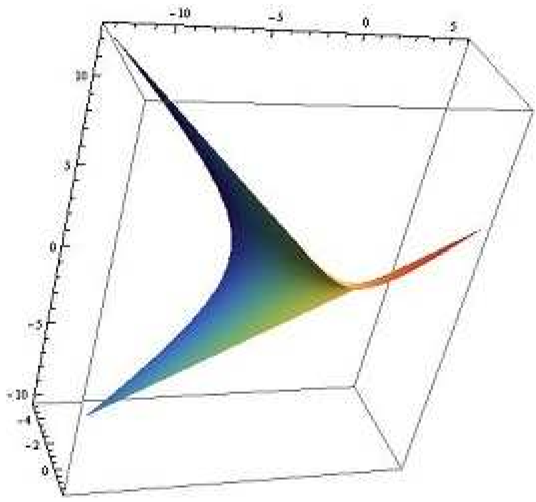

Figure 2. Null scroll $Y\left(s,t\right)$

Example 4.2. In $\mathbb{R}^5_2$, take a lightlike submanifold $M = X(x_1, x_2)$ as,

$$x_3 = \cos x_1, x_4 = \sin x_1, x_5 = x_2.$$

Then $RadTM = TM = Span\left\{\xi_1, \xi_2\right\}$ and $TM^{\perp} = Span\left\{\xi_1, W_1, W_2\right\}$ where,

$$\xi_1 = \partial_2 + \partial_5$$

$$\xi_2 = \partial_1 - \sin x_1 \partial_3 + \cos x_1 \partial_4$$
$$W_1 = -\sin x_1 \partial_1 + \partial_3$$
$$W_2 = \cos x_1 \partial_1 + \partial_4.$$

Therefore, $ltrTM = Span\left\{N_1, N_2\right\}$ where

$$N_1 = \frac{1}{2}\left[-\partial_2 + \partial_5\right] \text{ and } N_2 = \frac{1}{2}\left[-\partial_1 - \sin x_1 \partial_3 + \cos x_1 \partial_4\right].$$

Since $\left\{W_1, W_2\right\}$ are spacelike, it is the Case 1. Now, choose the unit timelike vector field

$$n^T = \frac{-\xi_1 + N_1}{\|-\xi_1 + N_1\|} = -\frac{3}{2\sqrt{2}}\partial_2 - \frac{1}{2\sqrt{2}}\partial_5$$

Then, the vector field

$$U = \frac{\xi_2 + N_2}{\|\xi_2 + N_2\|} = \frac{1}{2\sqrt{2}}\partial_1 - \frac{3}{2\sqrt{2}}\sin x_1 \partial_3 + \frac{3}{2\sqrt{2}}\cos x_1 \partial_4$$

is spacelike and the subspace $W_p = Span\{U, W_1, W_2\}_p$ where $p = X(u)$. One can take a spacelike unit vector field $n^S \in S^{n-1}$ as

$$n^S = \cos\theta_1 U + \sin\theta_1 \cos\theta_2 W_1 + \sin\theta_1 \sin\theta_2 W_2.$$

Then $n^T + n^S$ is a lightlike transversal normal vector field and

$$Y(x_1, x_2, \theta_1, \theta_2, t) = X(x_1, x_2) + t\left(n^T + n^S\right)(x_1, x_2)$$

is a null scroll along M.

Example 4.3. In $\mathbb{R}^{14}_4$, take a lightlike submanifold $M = X(\mathbf{x})$ as,

$$x_1 = x_{14}$$
$$x_2 = -x_{13}$$
$$x_3 = x_{12}$$
$$x_7 = \sqrt{1-(x_8)^2}.$$

where

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}).$$

Then, $TM = Span\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}\}$ where

$$Z_1 = \partial_1 + \partial_{14}$$
$$Z_2 = \partial_2 - \partial_{13}$$
$$Z_3 = \partial_3 + \partial_{12}$$
$$Z_4 = \partial_4$$
$$Z_5 = \partial_5$$
$$Z_6 = \partial_6$$
$$Z_7 = -x_8\partial_7 + x_7\partial_8$$
$$Z_8 = \partial_9$$
$$Z_9 = \partial_{10}$$
$$Z_{10} = \partial_{11}$$

and

$$RadTM = Span\{Z_1, Z_2, Z_3\}$$
$$S(TM^\perp) = Span\{W = x_7\partial_7 + x_8\partial_8\}$$
$$ltr(TM) = Span\left\{N_1 = \frac{1}{2}\left[-\partial_1 + \partial_{14}\right], N_2 = \frac{1}{2}\left[-\partial_2 - \partial_{13}\right], N_3 = \frac{1}{2}\left[-\partial_3 + \partial_{12}\right]\right\}.$$

Now, choose the timelike unit normal and spacelike vector fields as,

$$n^T = \frac{Z_1 - N_1}{\|Z_1 - N_1\|} = \frac{3}{2\sqrt{2}}\partial_1 + \frac{1}{2\sqrt{2}}\partial_{14}$$
$$U_1 = \frac{Z_2 + N_2}{\|Z_2 + N_2\|} = \frac{1}{2\sqrt{2}}\partial_2 - \frac{3}{2\sqrt{2}}\partial_{13}$$
$$U_2 = \frac{Z_3 + N_3}{\|Z_3 + N_3\|} = \frac{1}{2\sqrt{2}}\partial_3 + \frac{3}{2\sqrt{2}}\partial_{12}$$

Not that, $\langle U_1, U_2\rangle = 0$, $\langle U_1, W\rangle = \langle U_2, W\rangle = 0$. Therefore, it can be written that

$$n^S = \cos\theta_1 U_1 + \sin\theta_1 \cos\theta_2 U_2 + \sin\theta_1 \sin\theta_2 W.$$

Hence,

$$Y(\mathbf{x}, \theta_1, \theta_2, t) = \left[X + t\left(n^T + n^S\right)\right](\mathbf{x})$$

is a null scroll along M.

5. Conclusion

In this study, the theory of null scrolls is generalized with the proposed method. The method provides defining null scrolls in the higher dimensions and arbitrary indexes. Also, with the help of the base lightlike submanifold, one can determine the intrinsic objects of the null scrolls. For example, a relation between the second fundamental forms of the base lightike submanifold and the null scrolls, is given. Moreover, some geometric properties of the null scrolls, such as the flatness, being totally umbilic and totally geodesic, are investigated. Lastly, the constructed theory is supported by several examples.

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