Classification of Rectifying Space-Like Submanifolds in Pseudo-Euclidean Spaces

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ABSTRACT

The notions of rectifying subspaces and of rectifying submanifolds were introduced in [B.-Y. Chen, Int. Electron. J. Geom 9 (2016), no. 2, 1–8]. More precisely, a submanifold in a Euclidean $m$-space $\mathbb{E}^m$ is called a rectifying submanifold if its position vector field always lies in its rectifying subspace. Several fundamental properties and classification of rectifying submanifolds in Euclidean space were obtained in [B.-Y. Chen, op. cit.].

In this present article, we extend the results in [B.-Y. Chen, op. cit.] to rectifying space-like submanifolds in a pseudo-Euclidean space with arbitrary codimension. In particular, we completely classify all rectifying space-like submanifolds in an arbitrary pseudo-Euclidean space with codimension greater than one.

Keywords: Rectifying submanifold; rectifying subspace; pseudo-Euclidean space; concurrent vector field; space-like submanifold; position vector field.

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1. Introduction

Let $\mathbb{E}^3$ denote the Euclidean 3-space with its inner product $\langle \ , \ \rangle$. Consider a unit-speed space curve $x : I \to \mathbb{E}^3$, where $I = (\alpha, \beta)$ is a real interval. Let $x$ denote the position vector field of $x$ and $x'$ be denoted by $t$.

It is possible, in general, that $t'(s) = 0$ for some $s$; however, we assume that this never happens. Then we can introduce a unique vector field $n$ and positive function $\kappa$ so that $t' = \kappa n$. We call $t'$ the curvature vector field, $n$ the principal normal vector field, and $\kappa$ the curvature of the curve. Since $t$ is of constant length, $n$ is orthogonal to $t$. The binormal vector field is defined by $b = t \times n$, which is a unit vector field orthogonal to both $t$ and $n$. One defines the torsion $\tau$ by the equation $b' = -\tau n$.

The famous Frenet-Serret equations are given by

$$
\begin{align*}
t' &= \kappa n \\
n' &= -\kappa t + \tau b \\
b' &= -\tau n.
\end{align*}
$$

At each point of the curve, the planes spanned by $\{t, n\}$, $\{t, b\}$, and $\{n, b\}$ are known as the osculating plane, the rectifying plane, and the normal plane, respectively.

From elementary differential geometry it is well known that a curve in $\mathbb{E}^3$ lies in a plane if its position vector lies in its osculating plane at each point, and it lies on a sphere if its position vector lies in its normal plane at each point. A curve in the Euclidean 3-space is called a rectifying curve if its position vector field always lies in its rectifying plane (cf. [3]). Rectifying curves have many interesting properties. Such curves have been studied by many authors, see for instance, [1, 3, 10, 9, 13, 14, 15] among many others.

In [6], the first author introduced the notion of rectifying subspaces for Euclidean submanifolds. As a natural extension of rectifying curves, the first author defined the notion of rectifying submanifolds as Euclidean submanifolds whose position vector field always lies in its rectifying subspace [6]. Many fundamental properties of rectifying submanifolds are obtained in [6, 7]. In particular, the first author proved that a Euclidean
submanifold is rectifying if and only if the tangential component of its position vector field is a concurrent vector field. Furthermore, he completely determined rectifying submanifolds in a Euclidean space with arbitrary codimension.

In this article we extend the results of [6] to rectifying space-like submanifolds in a pseudo-Euclidean space with arbitrary codimension as a supplement to [6]. In particular, we completely classify all rectifying space-like submanifolds in an arbitrary pseudo-Euclidean space.

2. Preliminaries

For general references on submanifolds in pseudo-Riemannian manifolds, we refer to [5, 8, 16]. Let \( E^n_i \) denote the pseudo-Euclidean \( n \)-space equipped with the canonical pseudo-Euclidean metric \( g_0 \) of index \( i \) given by

\[
g_0 = - \sum_{r=1}^{i} du_r^2 + \sum_{t=i+1}^{m} du_t^2,
\]

where \((u_1, \ldots, u_m)\) is a rectangular coordinate system of \( E^n_i \).

Let \( x : M \to E^n_i \) be an isometric immersion of a pseudo-Riemannian \( n \)-manifold \( M \) into \( E^n_i \). For a point \( p \in M \), we denote by \( T_pM \) and \( T^\perp_p M \) the tangent and the normal spaces at \( p \). There is a natural orthogonal decomposition:

\[
T_p E^n_i = T_p M \oplus T^\perp_p M.
\]

Denote by \( \nabla \) and \( \tilde{\nabla} \) the Levi-Civita connections of \( M \) and \( E^n_i \), respectively. The formulas of Gauss and Weingarten are given respectively by

\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),
\]

\[
\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi
\]

for vector fields \( X, Y \) tangent to \( M \) and \( \xi \) normal to \( M \), where \( h \) is the second fundamental form, \( D \) the normal connection, and \( A \) the shape operator of \( M \).

For a given point \( p \in M \), the first normal space, of \( M \) in \( E^n_i \), denoted by \( \text{Im} h_p \), is the subspace defined by

\[
\text{Im} h_p = \text{Span}\{ h(X, Y) : X, Y \in T_p M \}.
\]

For each normal vector \( \xi \) at \( p \), the shape operator \( A_\xi \) is an endomorphism of \( T_p M \). The second fundamental form \( h \) and the shape operator \( A \) are related by

\[
\langle A_\xi X, Y \rangle = \langle h(X, Y), \xi \rangle,
\]

where \( \langle , \rangle \) denotes the scalar product on \( M \) as well as on the ambient space.

The equation of Gauss of \( M \) in \( E^n_i \) is given by

\[
R(X, Y; Z, W) = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle
\]

for \( X, Y, Z, W \) tangent to \( M \), where \( R \) denotes the curvature tensors of \( M \).

The covariant derivative \( \nabla h \) of \( h \) with respect to the connection on \( TM \oplus T^\perp M \) is defined by

\[
(\tilde{\nabla}_X h)(Y, Z) = D_X (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).
\]

The equation of Codazzi is

\[
(\tilde{\nabla}_X h)(Y, Z) = (\tilde{\nabla}_Y h)(X, Z).
\]

It follows from the definition of a rectifying curve \( x : I \to E^n \) that the position vector field \( x \) of \( x \) satisfies

\[
x(s) = \lambda(s) t(s) + \mu(s) b(s)
\]

for some functions \( \lambda \) and \( \mu \).
For a curve \( x : I \to \mathbb{E}^3 \) with \( n(s_0) \neq 0 \) at \( s_0 \in I \), the first normal space at \( s_0 \) is the line spanned by the principal normal vector \( n(s_0) \). Hence, the rectifying plane at \( s_0 \) is nothing but the plane orthogonal to the first normal space at \( s_0 \). Therefore, for a submanifold \( M \) of \( \mathbb{E}^n \) and a point \( p \in M \), we call the subspace of \( T_p \mathbb{E}^n \), orthogonal complement to the first normal space \( \text{Im} \ h_p \), the rectifying space of \( M \) at \( p \) (see [6]).

We make the following definition as in [6].

**Definition 2.1.** A pseudo-Riemannian submanifold \( M \) of a pseudo-Euclidean space \( \mathbb{E}^n \) is called a rectifying submanifold if the position vector field \( x \) of \( M \) always lies in its rectifying space. In other words, \( M \) is a rectifying submanifold if and only if

\[
\langle x(p), \text{Im} \ h_p \rangle = 0
\]

holds at every \( p \in M \).

### 3. Lemmas

A tangent vector \( v \) of a pseudo-Riemannian manifold \( \tilde{M}^m \) is called space-like (respectively, time-like) if \( v = 0 \) or \( \langle v, v \rangle > 0 \) (respectively, \( \langle v, v \rangle < 0 \)). A vector \( v \) is called light-like or null if \( v \neq 0 \) and \( \langle v, v \rangle = 0 \).

The light cone \( LC \) of \( \mathbb{E}^n \) is defined by

\[
LC = \{ v \in \mathbb{E}^n : \langle v, v \rangle = 0 \}.
\]  

Let \( r \) be a positive number. We put

\[
S^i_k(r^2) = \{ x \in \mathbb{E}^{k+1} : \langle x, x \rangle = r^2 \}, \quad i > 0,
\]

\[
H^k_i(-r^2) = \{ x \in \mathbb{E}^{k+1} : \langle x, x \rangle = -r^2 \}, \quad i > 0,
\]

\[
H^k_i(c) = \{ x \in \mathbb{E}^{k+1} : \langle x, x \rangle = -r^2 \quad \text{and} \quad x_1 > 0 \},
\]

\[
S^i_k(r^2) \quad \text{(respectively, } H^k_i(-r^2)) \text{ is a pseudo-Riemannian manifolds of curvature } 1/r^2 \text{ (respectively, } -1/r^2) \text{ with index } i. \text{ The } S^i_k(r^2) \text{ (respectively, } H^k_i(-r^2)) \text{ is known as a pseudo-sphere (respectively, pseudo-hyperbolic space).}
\]

The pseudo-Riemannian manifolds \( \mathbb{E}^k_1, S^i_k(c), H^k_i(c) \) are the standard models of Lorentzian space forms. In particular, \( \mathbb{E}^k_1, S^i_k(c), H^k_1(c) \) are the standard models of indefinite real space forms.

A submanifold \( M \) of \( \mathbb{E}^m \) is called space-like if each tangent vector of \( M \) is space-like.

By a cone in \( \mathbb{E}^m \) with vertex at the origin \( o \in \mathbb{E}^m \) we mean a ruled submanifold generated by a family of half lines initiated at \( o \). A submanifold of \( \mathbb{E}^m \) is called a conic submanifold with vertex at \( o \) if it is an open portion of a cone with vertex at \( o \).

For a space-like submanifold \( M \) of \( \mathbb{E}^m \), there exists a natural orthogonal decomposition of the position vector field \( x \) at each point; namely,

\[
x = x^T + x^N,
\]

where \( x^T \) and \( x^N \) denote the tangential and normal components of \( x \), respectively.

We put

\[
|x^T|^2 = \langle x^T, x^T \rangle, \quad |x^N|^2 = \langle x^N, x^N \rangle.
\]

**Lemma 3.1.** Let \( M \) be a pseudo-Riemannian submanifold of the pseudo-Euclidean space \( \mathbb{E}^m \). If the position vector field \( x \) of \( M \) in \( \mathbb{E}^m \) is either space-like or time-like, then \( x = x^T \) holds identically if and only if \( M \) is a conic submanifold with the vertex at the origin.

**Proof.** Let \( M \) be a pseudo-Riemannian submanifold of \( \mathbb{E}^m \). Assume that the position vector field \( x \) of \( M \) in \( \mathbb{E}^m \) is either space-like or time-like. If \( x = x^T \) holds identically, then \( e_1 = x/|x| \) is a unit vector field.

Put \( x = \rho e_1 \). Then we get

\[
\tilde{\nabla}_{e_1} x = e_1, \quad \tilde{\nabla}_{e_1} x = (e_1 \rho)e_1 + \rho \tilde{\nabla}_{e_1} e_1.
\]

Since \( \tilde{\nabla}_{e_1} e_1 \) is perpendicular to \( e_1 \), we find from (3.6) that \( \tilde{\nabla}_{e_1} e_1 = 0 \). Therefore the integral curves of \( e_1 \) are some open portions of generating lines in \( \mathbb{E}^m \). Moreover, because \( x = x^T \), the generating lines given by the integral curves of \( e_1 \) pass through the origin. Consequently, \( M \) is a conic submanifold with the vertex at the origin.

The converse is clear. \( \square \)
We recall the following definition of concurrent vector fields.

**Definition 3.1.** A non-trivial vector field \( C \) on a Riemannian (or more generally, on a pseudo-Riemannian) manifold \( M \) is called a concurrent vector field if it satisfies

\[
\nabla_X C = X
\]

(3.7)

for any vector \( X \) tangent to \( M \), where \( \nabla \) is the Levi-Civita connection of \( M \).

**Remark 3.1.** Since the position vector field of the pseudo-Euclidean space \( \mathbb{E}^m_\mathbb{E} \) is a concurrent vector field, it follows that the position vector field \( \mathbf{x} \) of any pseudo-Riemannian submanifold \( M \) in \( \mathbb{E}^m_\mathbb{E} \) satisfies

\[
\nabla_Z \mathbf{x} = Z \tag{3.8}
\]

for any \( Z \in TM \), where \( \nabla \) is the Levi-Civita connection of \( \mathbb{E}^m_\mathbb{E} \).

**Lemma 3.2.** Let \( M \) be a pseudo-Riemannian submanifold of \( \mathbb{E}^m_\mathbb{E} \). If the position vector field \( \mathbf{x} \) is either space-like or time-like, then the position vector field \( \mathbf{x} \) of \( M \) satisfies \( \mathbf{x} = \mathbf{x}^N \) identically if and only if \( M \) lies in one of the following hypersurfaces of \( \mathbb{E}^m_\mathbb{E} \):

1. a pseudo-sphere \( S_i^{m-1}(c^2) \); or
2. a pseudo-hyperbolic space \( H_i^{m-1}(-c^2) \) whenever \( i > 1 \); or
3. a hyperbolic space \( H_i^m(-c^2) \) whenever \( i = 1 \),

where \( c \) is a positive number.

**Proof.** Let \( x : M \rightarrow \mathbb{E}^{m}_\mathbb{E} \) be an isometric immersion of a pseudo-Riemannian \( n \)-manifold into \( \mathbb{E}^m_\mathbb{E} \) with space-like or time-like position vector field. If \( x = x^N \) holds identically, then we get from (3.8) that

\[
Z \langle x, x \rangle = 2 \langle \nabla_z x, x \rangle = 2 \langle Z, x^N \rangle = 0
\]

for any \( Z \in TM \). Thus \( M \) lies in one of the three hypersurfaces of \( \mathbb{E}^m_\mathbb{E} \).

The converse is easy to verify. \( \square \)

In views of Lemma 3.1 and Lemma 3.2 we make the following.

**Definition 3.2.** A rectifying submanifold \( M \) of \( \mathbb{E}^m_\mathbb{E} \) is called proper if its position vector field \( \mathbf{x} \) satisfies \( \mathbf{x} \neq \mathbf{x}^T \) and \( \mathbf{x} \neq \mathbf{x}^N \) at every point on \( M \).

In this article, we are only interested on proper rectifying submanifolds of \( \mathbb{E}^m_\mathbb{E} \) in views of Lemma 3.1 and Lemma 3.2.

For the proof of our main theorem we also need the following lemma.

**Lemma 3.3.** Let \( M \) be a pseudo-Riemannian submanifold of \( \mathbb{E}^m_\mathbb{E} \). If \( M \) is proper rectifying, then \( \langle x^N, x^N \rangle \) is constant on \( M \).

**Proof.** Let \( x : M \rightarrow \mathbb{E}^m_\mathbb{E} \) be an isometric immersion of a Riemannian \( n \)-manifold into \( \mathbb{E}^m_\mathbb{E} \). Consider the orthogonal decomposition

\[
x = x^T + x^N \tag{3.9}
\]

of the position vector field \( x \) of \( M \) in \( \mathbb{E}^m_\mathbb{E} \). It follows from (3.9) and the formula of Gauss and the formula of Weingarten that

\[
Z = \nabla_Z x = \nabla_Z x^T + h(Z, x^T) - A_{x^N} Z + D_{x^N} x^N
\]

(3.10)

for any \( Z \in TM \). By comparing the normal components in (3.10), we find

\[
D_{x^N} x^N = -h(Z, x^T).
\]

(3.11)

Therefore we obtain

\[
Z \langle x^N, x^N \rangle = 2 \langle D_{x^N} x^N, x^N \rangle = -\langle h(Z, x^T), x \rangle = 0,
\]

(3.12)

where we have used (2.11) in Definition 2.1. Since (3.12) holds identically for any \( Z \in TM \), we conclude that \( \langle x^N, x^N \rangle \) is constant on \( M \). \( \square \)

**Remark 3.2.** A submanifold \( M \) of \( \mathbb{E}^m_\mathbb{E} \) is called a \( T \)-submanifold (respectively, \( N \)-submanifold) if its position vector field \( x \) satisfies \( \langle x^T, x^T \rangle = \text{constant} \) (respectively, \( \langle x^N, x^N \rangle = \text{constant} \)) (cf. [2, 4]). Obviously, Lemma 3.3 implies that every proper rectifying pseudo-Riemannian submanifold of \( \mathbb{E}^m_\mathbb{E} \) is an \( N \)-submanifold.
4. Characterization of rectifying submanifolds in $\mathbb{E}^m_1$

The following result provides a very simple characterization of rectifying submanifolds.

**Theorem 4.1.** If the position vector field $x$ of a pseudo-Riemannian submanifold $M$ in $\mathbb{E}^m_1$ satisfies $x^N \neq 0$, then $M$ is a proper rectifying submanifold if and only if $x^T$ is a concurrent vector field on $M$.

**Proof.** Let $M$ be a space-like submanifold of $\mathbb{E}^m_1$. Then (3.10) holds. After comparing the tangential components in (3.10), we obtain

$$A_{x^N} Z = \nabla_Z x^T - Z. \tag{4.1}$$

Assume that $M$ is a proper rectifying submanifold of $\mathbb{E}^m_1$. Then we have $x^T \neq 0$ and $x^N \neq 0$. Moreover, it follows from the Definition 2.1 that

$$\langle A_{x^N} X, Y \rangle = (x, h(X, Y)) = 0 \tag{4.2}$$

for $X, Y \in TM$. Since $M$ is space-like, we find from (4.1) that $A_{x^N} = 0$. Therefore (3.8) yields

$$\nabla_Z x^T = Z, \tag{4.3}$$

for any $Z \in TM$. Consequently, $x^T$ is a concurrent vector field on $M$.

Conversely, if $x^T$ is a concurrent vector field on $M$, then (3.7) and (4.1) give $A_{x^N} = 0$. Therefore we obtain (4.3). Consequently, $M$ is a proper rectifying submanifold due to $x^N \neq 0$ by assumption. \hfill $\blacksquare$

The next result shows that every proper rectifying space-like submanifold is a warped product.

**Theorem 4.2.** Let $M$ be a proper rectifying space-like submanifold $M$ of $\mathbb{E}^m_1$. Then $M$ is a warped product manifold $I \times_s F$ with warping metric

$$g = ds^2 + s^2 g_F, \tag{4.4}$$

such that $x^T = s \partial/\partial s$ and $g_F$ is the metric tensor of a Riemannian manifold $F$.

**Proof.** Let $M$ be a proper rectifying space-like submanifold of $\mathbb{E}^m_1$. Then we have $x^T \neq 0$ and $x^N \neq 0$. Thus we may put

$$x^T = \rho e_1, \quad \rho = |x^T| > 0, \tag{4.5}$$

where $e_1$ is a space-like unit vector field. We may extend $e_1$ to a local orthonormal frame $e_1, e_2, \ldots, e_n$ on $M$.

Obviously, it follows from (4.5) that $\rho = (x, e_1)$. Thus, by taking the derivative of $\rho$ with respect to $e_j$ for $j = 1, \ldots, n$ and using (2.3) and (3.8), we find

$$e_j \rho = \delta_{1j} + (x, h(e_1, e_j)), \tag{4.6}$$

where $\delta_{ij} = 1$ or 0 depending on $i = j$ or $i \neq j$. Combining (2.11) and (4.6) gives

$$e_1 \rho = 1, \quad e_2 \rho = \cdots = e_n \rho = 0.$$

Therefore we get $\rho = \rho(s)$ and $\rho'(s) = 1$, which imply $\rho(s) = s + b$ for some real number $b$. Hence, after applying a suitable translation on $s$ if necessary, we have $\rho = s$. Therefore, we obtain

$$x^T = se_1 = s \frac{\partial}{\partial s}. \tag{4.7}$$

Since $M$ is a proper rectifying space-like submanifold, Theorem 4.1 implies that $x^T = se_1$ is a concurrent vector field. Thus we find from (4.3) that

$$e_1 = \nabla_{e_1} x^T = \nabla_{e_1} se_1 = e_1 + s \nabla_{e_1} e_1, \tag{4.8}$$

which implies $\nabla_{e_1} e_1 = 0$. Therefore the integral curves of $e_1$ are geodesics of $M$. Consequently, the distribution $\mathcal{D}_1$ spanned by $e_1$ is a totally geodesic foliation.
From (4.3) we also find
\[ e_i = \nabla e_i x^T = s \nabla e_i e_1, \quad i = 2, \ldots, n, \] (4.9)
which gives
\[ \omega^i_j(e_i) = \frac{\delta_{ij}}{s}, \quad i, j = 2, \ldots, n. \] (4.10)
We conclude from (4.10) that the distribution \( \mathcal{D} \) is integrable whose leaves are totally umbilical hypersurfaces of \( M \). Moreover, it follows from (4.10) that the mean curvature of leaves of \( \mathcal{D} \) are given by \( s^{-1} \). Since the leaves of \( \mathcal{D} \) are hypersurfaces, it follows that the mean curvature vector field of the leaves of \( \mathcal{D} \) is parallel in the normal bundle in \( M \). Therefore the distribution \( \mathcal{D} \) is a spherical foliation. Consequently, by applying a result of [12] (or Theorem 4.4 of [5, page 90]) we conclude that \( M \) is locally a warped product \( I \times_s F \), where \( F \) is a Riemannian \( (n - 1) \)-manifold. Therefore the metric tensor \( g \) of \( M \) takes the form (4.4).

5. Main result

The main result of this article is the following classification theorem.

**Theorem 5.1.** Let \( M \) be a proper rectifying space-like submanifold of the pseudo-Euclidean \( n \)-space \( \mathbb{E}^m_1 \) with index \( i > 0 \). If \( \text{codim} \, M \geq 2 \), then one of the following four cases occurs:

(a) There exist a positive number \( c \) and local coordinate systems \( \{s, u_2, \ldots, u_n\} \) on \( M \) such that the immersion of \( M \) in \( \mathbb{E}^m_1 \) is given by
\[ x(s, u_2, \ldots, u_n) = \sqrt{s^2 + c^2} \, Y(s, u_2, \ldots, u_n), \] (5.1)
where \( Y = Y(s, u_2, \ldots, u_n) \) defines a space-like submanifolds of the unit pseudo-sphere \( S_i^{m-1}(1) \subset \mathbb{E}^m_1 \) such that the induced metric \( g_Y \) of \( Y \) is given by
\[ g_Y = \frac{c^2}{(s^2 + c^2)^2} ds^2 + \frac{s^2}{s^2 + c^2} \sum_{j,k=2}^n g_{jk}(u_2, \ldots, u_n) du_jdu_k. \] (5.2)

(b) There exist local coordinate systems \( \{s, u_2, \ldots, u_n\} \) on \( M \) such that the immersion of \( M \) in \( \mathbb{E}^m_1 \) is given by
\[ x(s, u_2, \ldots, u_n) = s W(s, u_2, \ldots, u_n), \quad s \neq 0, \] (5.3)
where \( W = W(s, u_2, \ldots, u_n) \) lies in the unit pseudo-sphere \( S_i^{m-1}(1) \subset \mathbb{E}^m_1 \) such that \( W_s \) is a light-like normal vector field of \( M \) and the induced metric tensor of \( W \) is of the following degenerate form:
\[ g_W = \sum_{j,k=2}^n g_{jk}(u_2, \ldots, u_n) du_jdu_k \] (5.4)
with positive definite \( (g_{jk}) \), \( j, k = 2, \ldots, n \).

(c) There exist a positive number \( c \) and local coordinate systems \( \{s, u_2, \ldots, u_n\} \) on \( M \) such that the immersion of \( M \) in \( \mathbb{E}^m_1 \) is given by
\[ x(s, u_2, \ldots, u_n) = \sqrt{s^2 - c^2} \, U(s, u_2, \ldots, u_n), \quad s^2 > c^2, \] (5.5)
where \( U = U(s, u_2, \ldots, u_n) \) lies in the unit pseudo-sphere \( S_i^{m-1}(1) \subset \mathbb{E}^m_1 \) such that the induced metric \( g_U \) of \( U \) is given by
\[ g_U = \frac{c^2}{(s^2 - c^2)^2} ds^2 + \frac{s^2}{s^2 - c^2} \sum_{j,k=2}^n g_{jk}(u_2, \ldots, u_n) du_jdu_k. \] (5.6)

(d) There exist a positive number \( c \) and local coordinate systems \( \{s, u_2, \ldots, u_n\} \) on \( M \) such that the immersion of \( M \) in \( \mathbb{E}^m_1 \) is given by
\[ x(s, u_2, \ldots, u_n) = \sqrt{c^2 - s^2} \, V(s, u_2, \ldots, u_n), \quad c^2 > s^2, \] (5.7)
where \( V = V(s, u_2, \ldots, u_n) \) lies in the pseudo-hyperbolic space \( H_i^{m-1}(-1) \subset \mathbb{E}^m_1 \) for \( i > 1 \) (respectively, hyperbolic space \( H_i^{m-1}(-1) \subset \mathbb{E}^m_1 \) for \( i = 1 \)) such that the induced metric \( g_V \) of \( V \) is given by
\[ g_V = \frac{c^2}{(s^2 - c^2)^2} ds^2 + \frac{s^2}{c^2 - s^2} \sum_{j,k=2}^n g_{jk}(u_2, \ldots, u_n) du_jdu_k. \] (5.8)
Conversely, each of the four cases above gives rise to a proper rectifying space-like submanifold of $\mathbb{E}^m_i$.

Proof. Assume that $M$ is a proper rectifying space-like submanifold of $\mathbb{E}^m_i$ with $m \geq 2 + \dim M$. Then we have $x^T \neq 0$ and $x^N \neq 0$. Thus we may put

$$x^T = \rho e_1, \quad \rho = |x^T| > 0,$$

(5.9)

where $e_1$ is a space-like unit vector field. We may extend $e_1$ to a local orthonormal frame $e_1, e_2, \ldots, e_n$ on $M$. Clearly, we have $\langle x, e_j \rangle = 0$ for $j = 2, \ldots, n$.

Define the connection forms $\omega^j_i, i, j = 1, \ldots, n$, by

$$\nabla_x e_i = \sum_{j=1}^n \omega^j_i (x)e_j, \quad i = 1, \ldots, n,$$

(5.10)

where $\nabla$ is the Levi-Civita connection of $M$.

For $j, k = 2, \ldots, n$, we find

$$0 = e_k (x, e_j) = \delta_{jk} + \langle x, \nabla_{e_k} e_j \rangle + \langle x, h(e_j, e_k) \rangle = \delta_{jk} + \langle x, \nabla_{e_k} e_j \rangle,$$

(5.11)

where we have applied (2.11) from Definition 2.1, (2.3) and (3.8).

Since $h(X, Y)$ is symmetric in $X$ and $Y$, we derive from (5.10) and (5.11) that

$$\omega^j_i (e_k) = \omega^k_i (e_j), \quad j, k = 2, \ldots, n.$$  

(5.12)

It follows from (5.10), (5.12) and the Frobenius theorem that the distribution $\mathcal{D}$ spanned by $e_2, \ldots, e_n$ is an integrable distribution.

On the other hand, the distribution $\mathcal{D}^\perp = \text{Span} \{e_1\}$ is also integrable since it is of rank one. Therefore, there exists a local coordinate system $\{s, u_2, \ldots, u_n\}$ on $M$ such that

$$e_1 = \frac{\partial}{\partial s} \text{ and } \mathcal{D} = \text{Span} \left\{ \frac{\partial}{\partial u_2}, \ldots, \frac{\partial}{\partial u_n} \right\}.$$  

Obviously, it follows from (5.9) that $\rho = \langle x, e_1 \rangle$. Now, by taking the derivative of $\rho$ with respect to $e_j$ for $j = 1, \ldots, n$ and using (2.3) and (3.8), we find

$$e_j \rho = \delta_{1j} + \langle x, h(e_1, e_j) \rangle.$$  

(5.13)

After combining (2.11) and (5.13) we find $e_1 \rho = 1$ and $e_2 \rho = \cdots = e_n \rho = 0$. Therefore we have

$$\rho = \rho(s), \quad \rho'(s) = 1$$

which imply

$$\rho(s) = s + b.$$  

(5.14)

for some real number $b$. Consequently, after applying a suitable translation on $s$ if necessary, we obtain $\rho = s$. Consequently, (5.9) implies that the position vector field satisfies

$$x = s e_1 + x^N.$$  

(5.15)

Moreover, since $M$ is a proper rectifying submanifold, Lemma 3.3 implies that $\langle x^N, x^N \rangle$ is constant on $M$. Therefore we find

$$\langle x, x \rangle = \begin{cases} 
  s^2 + c^2, & \text{if } \langle x^N, x^N \rangle > 0, \\
  s^2, & \text{if } \langle x^N, x^N \rangle = 0, \\
  s^2 - c^2, & \text{if } \langle x^N, x^N \rangle < 0,
\end{cases}$$  

(5.16)

where $c$ is a positive number.

Now, we divide the proof of the theorem into three cases.

Case (1): $\langle x, x \rangle = s^2 + c^2$ with $c > 0$. In this case, we may put

$$x(s, u_2, \ldots, u_n) = \sqrt{s^2 + c^2} Y(s, u_2, \ldots, u_n),$$  

(5.17)
for some $\mathbb{E}^m$-valued function $Y = Y(s, u_2, \ldots, u_n)$ satisfying $\langle Y, Y \rangle = 1$. Therefore the image of $Y$ lies in the pseudo-sphere $S^{m-1}_i(1) \subset \mathbb{E}^{m-1}$. It follows from (5.17) that

$$
\frac{\partial x}{\partial s} = \frac{s}{\sqrt{s^2 + c^2}} Y + \sqrt{s^2 + c^2} Y_s,
\frac{\partial x}{\partial u_j} = \sqrt{s^2 + c^2} Y_{u_j}, \ j = 2, \ldots, n.
$$

Using (5.18) together with the fact that $e_1 = \partial x/\partial s$ is a unit vector field orthogonal to the distribution $\mathcal{D}$, we derive that

$$
\langle Y_s, Y_s \rangle = c^2 (s^2 + c^2), \ \langle Y_s, Y_{u_j} \rangle = 0, \ j = 2, \ldots, n.
$$

Therefore the metric tensor $g_Y$ of $Y$ induced from $S^{m-1}_i(1)$ takes the following form:

$$
g_Y = \frac{c^2}{(s^2 + c^2)^2} ds^2 + \frac{s^2}{s^2 + c^2} \sum_{j,k=2}^n g_{jk}(s, u_2, \ldots, u_n) du_j du_k,
$$

where $(g_{jk})$ is positive definite. In particular, (5.17) and (5.20) show that the submanifold defined by $Y$ is also space-like.

Now, by applying (5.18) and (5.20) we know that the metric tensor $g$ of $M$ is of the form:

$$
g = ds^2 + s^2 \sum_{j,k=2}^n g_{jk}(s, u_2, \ldots, u_n) du_j du_k.
$$

After a straightforward long computation we find from (5.21) that the Levi-Civita connection of $M$ satisfies

$$
\nabla X \frac{\partial}{\partial s} = 0,
\nabla \frac{\partial}{\partial u_j} = \frac{1}{s} \frac{\partial}{\partial u_j} + \frac{1}{2} \sum_{k=2}^n \left( \sum_{t=2}^n g_{kt} \frac{\partial g_{jt}}{\partial s} \right) \frac{\partial}{\partial u_k}, \ j = 2, \ldots, n,
$$

where $(g^{jk})$ is the inverse matrix of $(g_{ij})$. Because $M$ is a proper rectifying space-like submanifold of $\mathbb{E}^m$, it follows from Theorem 4.1 that

$$
\nabla \frac{\partial}{\partial u_j} x^T = \frac{\partial}{\partial u_j}, \ j = 2, \ldots, n.
$$

Therefore, after applying (4.7), (5.22) and (5.23) we obtain

$$
\sum_{t=2}^n g^{kt} \frac{\partial g_{jt}}{\partial s} = 0, \ j, k = 2, \ldots, n.
$$

Because $(g^{jk})$ is positive definite, system (5.24) implies

$$
\frac{\partial g_{jk}}{\partial s} = 0, \ j, t = 2, \ldots, n.
$$

Therefore (5.31) must take the form of (5.4). Consequently, (5.20) reduces to (5.2).

Conversely, let us consider a space-like submanifold $M$ of $\mathbb{E}^m$ defined by (5.1) satisfying $\langle Y, Y \rangle = 1$ such that the metric tensor $g_Y$ is given by (5.2). Then we obtain (5.18) and (5.19) from (5.1). It follows from (5.2), (5.18) and (5.19) that the metric tensor $g$ of $M$ is given by

$$
g = ds^2 + s^2 \sum_{j,k=2}^n g_{jk}(u_2, \ldots, u_n) du_j du_k.
$$

Now, it is straightforward to verify from (5.25) that the Levi-Civita connection of $M$ satisfies

$$
\nabla X \frac{\partial}{\partial s} = 0, \ \nabla \frac{\partial}{\partial u_j} = \frac{1}{s} \frac{\partial}{\partial u_j}, \ j = 2, \ldots, n.
$$
Since \(\langle Y, Y \rangle = 1\), (5.1) implies \(\langle x, Y_u \rangle = 0\) for \(j = 2, \ldots, n\). Thus we find from (5.18) that
\[
\langle x, x_{u_j} \rangle = 0, \quad j = 2, \ldots, n.
\]
(5.27)

Therefore, we obtain \(x^T = s \frac{\partial}{\partial s}\). Now, by applying (5.26) it is easy to verify that \(x^T\) is a concurrent vector field on \(M\). Moreover, it is direct to show that the normal component of \(x\) is given by
\[
x^N = \frac{c^2}{\sqrt{s^2 + c^2}} Y - s \frac{\partial}{\partial s} Y_s,
\]
which is always non-zero everywhere on \(M\). Consequently, the immersion defined by case (a) gives rise to a proper rectifying space-like submanifold of \(E^n_m\).

Case (2): \(\langle x, x \rangle = s^2, s \neq 0\). In this case, \(x^N\) is a light-like normal vector field of \(M\).

We put
\[
x(s, u_2, \ldots, u_n) = s W(s, u_2, \ldots, u_n), \quad s \neq 0,
\]
(5.28)

for some \(\mathbb{E}^m\)-valued function \(W = W(s, u_2, \ldots, u_n)\) satisfying \(\langle W, W \rangle = 1\). Therefore the image of \(W\) lies in the pseudo-sphere \(S^{m-1}_i \subset \mathbb{E}^{m-1}_i\).

It follows from (5.28) that
\[
\frac{\partial x}{\partial s} = W + s W_s, \quad \frac{\partial x}{\partial u_j} = s W_{u_j}, \quad j = 2, \ldots, n.
\]
(5.29)

Using (5.29), \(\langle W, W \rangle = 1\) and the fact that \(e_1 = \partial x/\partial s\) is a unit vector field orthogonal to the distribution \(\mathcal{D}\), we derive that
\[
\langle W_s, W_s \rangle = 0, \quad \langle W_s, W_{u_j} \rangle = 0, \quad j = 2, \ldots, n.
\]
(5.30)

If we put \(g_{jk} = \langle W_{u_j}, W_{u_k} \rangle\), then it follows from (5.29) and (5.30) that the metric tensor \(g^W\) of \(W\) is a generate one given by
\[
g^W = \sum_{j,k=2}^n g_{jk}(s, u_2, \ldots, u_n) du_j du_k.
\]
(5.31)

Then it follows from (5.28) and (5.31) that the induced metric \(g\) of \(M\) is given by
\[
g = ds^2 + s^2 \sum_{j,k=2}^n g_{jk}(s, u_2, \ldots, u_n) du_j du_k.
\]
(5.32)

Since \(M\) is a proper rectifying space-like submanifold of \(\mathbb{E}^m_i\), it follows from Theorem 4.1 that \(x^T\) is a concurrent vector field. Therefore, we may apply the same argument as in Case (1) to conclude that \(\frac{\partial g_{jk}}{\partial s} = 0\) for \(j, t = 2, \ldots, n\). Therefore (5.31) must take the form of (5.4).

Conversely, let us consider an immersion \(x : M \to \mathbb{E}^m_i\) of a Riemannian \(n\)-manifold \(M\) into \(\mathbb{E}^m_i\) given by
\[
x(s, u_2, \ldots, u_n) = s W(s, u_2, \ldots, u_n), \quad \langle W, W \rangle = 1, \quad s \neq 0,
\]
(5.33)
such that \(W_s\) is a light-like normal vector field and the metric tensor of \(W\) is of the following degenerate form:
\[
g^W = \sum_{j,k=2}^n g_{jk}(s, u_2, \ldots, u_n) du_j du_k
\]
(5.34)

with positive definite matrix \((g_{jk})\), \(j, k = 2, \ldots, n\). Then it follows from (5.33) and (5.34) that the induced metric \(g\) of \(M\) is given by
\[
g = ds^2 + s^2 \sum_{j,k=2}^n g_{jk}(u_2, \ldots, u_n) du_j du_k.
\]
(5.35)

From (5.34) we get
\[
x_s = W + s W_s, \quad x_{u_j} = s W_{u_j}, \quad j = 2, \ldots, n.
\]
(5.36)
Thus we find from (5.33) and (5.36) that

\[ x = sx_s - s^2 W_s. \]  

Because \( W_s \) is a light-like normal vector field and \( x_s \) is tangent to \( M \), we obtain from (5.37) that

\[ x^T = sx_s \quad \text{and} \quad x^N = -s^2 W_s \neq 0. \]  

(5.38)

Now, we may derive from (5.35) and (5.38) as before that \( x^T \) is a concurrent vector field on \( M \). Consequently, \( M \) is a rectifying space-like submanifold of \( \mathbb{E}^m_n \) according to Theorem 4.1. This gives Case (b) of the theorem.

Case (3): \( \langle x, x \rangle = s^2 - c^2 \neq 0 \). By applying a method similar to Case (1), we will obtain either Case (c) or Case (d) according to \( s^2 > c^2 \) or \( s^2 < c^2 \), respectively.

References


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