Some Characterizations of Constant Ratio Curves According to Type-2 Bishop Frame in Euclidean 3-space $E^3$

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Abstract
In this paper, we study a twisted curve in the 3-dimensional Euclidean space $E^3$ as a curve whose position vector can be determined as linear combination of its type-2 Bishop frame. We research these curves according to their curvature functions. Moreover we obtain some results of $T$-constant and $N$-constant type curves in the 3-dimensional Euclidean space $E^3$.

Keywords: Position vector; type-2 Bishop frame; constant ratio curves.

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1. Introduction
The theory of curves has an important role in differential geometry. One of these curves is twisted curve, a curve $x : I \subset \mathbb{R} \rightarrow E^3$ which has non-zero Frenet curvatures $k_1(s)$ and $k_2(s)$ is called twisted curve. For a regular curve $x(s)$, the position vector of $x$ can be decomposed into its tangential and normal components at each point:

$$x = x^T + x^N. \tag{1.1}$$

A curve $x(s)$ with $k_1(s) > 0$ is called constant ratio if the ratio $\|x^T\| : \|x^N\|$ is constant on $x(I)$. Here $\|x^T\|$ and $\|x^N\|$ denote the length of $x^T$ and $x^N$, respectively [4]. A curve in $E^n$ is said to be $T$-constant (resp. $N$-constant) if the tangential component $x^T$ (resp. the normal component $x^N$) of its position vector $x$ is of constant length [4]. In recent years constant ratio curves are studied in Euclidean and Minkowski space [7, 3, 8, 2].

On the other hand, L.R. Bishop defined Bishop frame, which is known alternative or parallel frame of the curves with the help of parallel vector fields [1]. Then, S. Yılmaz and M. Turgut introduced a new version of the Bishop frame which is called type-2 Bishop frame [10]. Thereafter, E. Özyılmaz studied classical differential geometry of curves according to type-2 Bishop trihedra [9].

In this study we researched a twisted curve in the 3-dimensional Euclidean space $E^3$ as a curve whose position vector satisfies the following parametric equation

$$x(s) = \lambda(s) N_1(s) + \mu(s) N_2(s) + \gamma(s) B(s) \tag{1.2}$$

where $\lambda, \mu, \gamma$ are differentiable functions and $\{N_1, N_2, B\}$ is its type-2 Bishop frame. We characterize these curves according to their curvature functions. Moreover we obtain some results of $T$-constant and $N$-constant type curves in the 3-dimensional Euclidean space $E^3$.

2. Preliminaries
The standard flat metric of 3-dimensional Euclidean space $E^3$ is given by

$$\langle , \rangle : dx_1^2 + dx_2^2 + dx_3^2 \tag{2.1}$$
where \((x_1, x_2, x_3)\) is a rectangular coordinate system of \(E^3\). For an arbitrary vector \(x\) in \(E^3\), the norm of this vector is defined by \(\|x\| = \sqrt{\langle x, x \rangle}\). \(\alpha\) is called a unit speed curve, if \(\|\alpha', \alpha''\| = 1\). Suppose that \(\{t, n, b\}\) is the moving Frenet-Serret frame along the curve \(\alpha\) in \(E^3\). For a unit speed curve \(\alpha\), the Frenet-Serret formulae can be given as
\[
\begin{align*}
t' &= \kappa n \\
n' &= -\kappa t + \tau b \\
b' &= -\tau n
\end{align*}
\]
where
\[
\langle t, t \rangle = \langle n, n \rangle = \langle b, b \rangle = 1,
\langle t, n \rangle = \langle t, b \rangle = \langle n, b \rangle = 0.
\]
and here, \(\kappa = \kappa(s) = \|t'(s)\|\) and \(\tau = \tau(s) = -\langle n, b' \rangle\). Furthermore, the torsion of the curve \(\alpha\) can be given by
\[
\tau = \frac{[\alpha', \alpha'' \times \alpha''']}{\kappa^2}.
\]
Along the paper, we assume that \(\kappa \neq 0\) and \(\tau \neq 0\).

Bishop frame is an alternative approach to define a moving frame. Assume that \(\alpha(s)\) is a unit speed regular curve in \(E^3\). The type-2 Bishop frame of the \(\alpha(s)\) is expressed as \([10]\)
\[
\begin{align*}
N_1' &= -k_1 B, \\
N_2' &= -k_2 B, \\
B' &= k_1 N_1 + k_2 N_2.
\end{align*}
\]
The relation matrix may be expressed as
\[
\begin{bmatrix}
t \\n\theta \\
\end{bmatrix} =
\begin{bmatrix}
\sin \theta(s) & -\cos \theta(s) & 0 \\
\cos \theta(s) & \sin \theta(s) & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
N_1 \\
N_2 \\
B
\end{bmatrix}.
\]
where \(\theta(s) = \int_0^s \kappa(s) \, ds\). Then, type-2 Bishop curvatures can be defined in the following
\[
\begin{align*}
k_1(s) &= -\tau(s) \cos \theta(s), \\
k_2(s) &= -\tau(s) \sin \theta(s).
\end{align*}
\]
On the other hand,
\[
\theta' = \kappa = \frac{(k_2 k_1)'}{1 + \left(\frac{k_2}{k_1}\right)^2}.
\]
The frame \(\{N_1, N_2, B\}\) is properly oriented, \(\tau\) and \(\theta(s) = \int_0^s \kappa(s) \, ds\) are polar coordinates for the curve \(\alpha\). Then, \(\{N_1, N_2, B\}\) is called type-2 Bishop trihedra and \(k_1, k_2\) are called type-2 Bishop curvatures.

3. Constant Ratio Curves According to Type-2 Bishop Frame

Let \(x(s)\) be a twisted curve whose position vector can be determined as linear combination of its type-2 Bishop frame, then its position vector can be written as
\[
x(s) = \lambda(s) N_1(s) + \mu(s) N_2(s) + \gamma(s) B(s)
\]
where \(\lambda, \mu, \gamma\) are differentiable functions and \(\{N_1, N_2, B\}\) is its type-2 Bishop frame. Differentiating the equation (3.1) and using equation (2.3) we get
\[
x'(s) = (\lambda' + \gamma k_1) N_1(s) + (\mu' + \gamma k_2) N_2(s) + (\gamma' - \lambda k_1 - \mu k_2) B(s)
\]
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where \( k_1(s) \) and \( k_2(s) \) are Bishop curvatures. On the other hand if \( N_1 \) is taken instead of tangent vector, and considering above equation we have the following

\[
\begin{align*}
  \lambda' + \gamma k_1 - 1 &= 0 \\
  \mu' + \gamma k_2 &= 0 \\
  \gamma' - \lambda k_1 - \mu k_2 &= 0.
\end{align*}
\]

(3.2)

**Definition 3.1.** Let \( x : I \subset \mathbb{R} \to E^n \) be a unit speed curve in \( E^n \). Then the position vector of \( x \) can be decomposed into its tangential and normal components at each point as

\[
x = x^T + x^N
\]

if the ratio \( \|x^T\| : \|x^N\| \) is constant on \( x(I) \) then \( x \) is said to be constant ratio [4].

For a unit speed curve \( x \) in \( E^n \) the gradient of the distance function \( \rho = \|x(s)\| \) is given by

\[
\text{grad}\rho = \frac{dp}{ds} x'(s) = \frac{\langle x(s), x'(s) \rangle}{\|x(s)\|} x'(s)
\]

(3.3)

where \( T \) is the tangent vector of \( x \). The following results can be given for constant ratio curves.

**Theorem 3.1.** [5] Let \( x : I \subset \mathbb{R} \to E^n \) be a unit speed regular curve in \( E^n \). Then \( \|\text{grad}\rho\| = c \) holds for a constant \( c \) if and only if the following three cases occur:

(i) \( x(I) \) is contained in a hypersphere centered at the origin.

(ii) \( x(I) \) is an open portion of a line through the origin.

(iii) \( x(s) = csy(s), c \in (0,1), \) where \( y(y(u)) \) is a unit curve on the unit sphere of \( E^n \) centered at the origin and \( u = \sqrt{1 - c^2} \ln s \).

**Corollary 3.1.** [5] Let \( x : I \subset \mathbb{R} \to E^n \) be a unit speed regular curve in \( E^n \). Then up to a translation of the arc length function \( s \), we have

(i) \( \|\text{grad}\rho\| = 0 \iff x(I) \) is contained in a hypersphere centered at the origin.

(ii) \( \|\text{grad}\rho\| = 1 \iff x(I) \) is an open portion of a line through the origin.

(iii) \( \|\text{grad}\rho\| = c \iff \rho = \|x(s)\| = cs \) for \( c \in (0,1) \).

(iv) If \( n = 2 \) and \( \|\text{grad}\rho\| = c \) for \( c \in (0,1) \), then the curvature of \( x \) satisfies

\[
\kappa^2 = \frac{1 - c^2}{c^2s^2 + b},
\]

for some real constant \( b \).

For twisted curves according to type-2 Bishop frame in \( E^3 \) we get the following results.

**Proposition 3.1.** Let \( x : I \subset \mathbb{R} \to E^3 \) be a unit speed curve in \( E^3 \). If \( x \) is a curve of constant ratio then its position vector can be written as

\[
x(s) = (c^2 s) N_1(s) - \left[ \frac{c^2 k_1^2}{k_2} + \frac{(1 - c^2) k_1}{k_1k_2} \right] N_1(s) + \left( \frac{1 - c^2}{k_1} \right) B(s)
\]

(3.4)
Proof. Let \( x \) be a curve of constant ratio, then from corollary 3.1. the distance function of \( x \) can be written as 
\[ \rho = \| x(s) \| = cs \]
for some real constant \( c \). Moreover considering (3.3) we have 
\[ \| \text{grad} \rho \| = \frac{\langle x(s), x'(s) \rangle}{\| x(s) \|} \]
Because of \( x \) is a twisted curve of \( E^3 \), the equation (3.1) is satisfied. Then we get 
\[ \lambda = c^2 s. \]
Therefore substituting \( \lambda = c^2 s \) in the equation (3.2) we obtain
\begin{align*}
\mu &= -\frac{c^2 s k_1}{k_2} - \frac{(1 - c^2) k'_1 k_2}{k_1^2 k_2}, \\
\gamma &= 1 - \frac{c^2}{k_1}.
\end{align*}
If we consider the above value of \( \lambda, \mu, \gamma \) and substituting these value in equation (3.1) we obtain the equation (3.4) which complete the proof.

4. \( T \)-Constant Curves

Definition 4.1. Let \( x : I \subset \mathbb{R} \to E^n \) be a unit speed curve in \( E^n \). If \( \| x^T \| \) is constant then \( x \) is called a \( T \)-constant curve. For a \( T \)-constant curve \( x \) either \( \| x^T \| = 0 \) or \( \| x^T \| = \eta \) for some non-zero smooth function \( \eta \). Moreover, a \( T \)-constant curve \( x \) is called first kind if \( \| x^T \| = 0 \), otherwise second kind [6].

As a result of the equation (3.2), we obtain the following expression.

Theorem 4.1. Let \( x : I \subset \mathbb{R} \to E^3 \) be a unit speed twisted curve in \( E^3 \) that satisfies the equation (3.1). Then \( x \) is a \( T \)-constant curve of first kind if and only if
\[ \frac{k_2}{k_1} - \left( \frac{k'_1}{k_1^2 k_2} \right)' = 0 \]
where \( k_1, k_2 \) are Bishop curvatures.

Proof. Suppose that \( x \) is a \( T \)-constant curve of first kind. Then using the first and third equation of (3.2) we get
\begin{align*}
\gamma &= \frac{1}{k_1}, \\
\mu &= -\frac{k'_1}{k_1^2 k_2},
\end{align*}
where \( k_1, k_2 \) are Bishop curvatures. Substituting above equation into the second equation of (3.2) we obtain the desired result.

Theorem 4.2. Let \( x : I \subset \mathbb{R} \to E^3 \) be a unit speed twisted curve in \( E^3 \) that satisfies the equation (3.1). If \( x \) is a \( T \)-constant curve of second kind then the position vector of the curve is given by
\[ x = \lambda N_1(s) - \left( \frac{\lambda k_1}{k_2} + \frac{k'_1}{k_1^2 k_2} \right) N_2(s) + \frac{1}{k_1} B(s) \]
where \( \lambda \) is a constant function.

Proof. Suppose that \( x \) is a \( T \)-constant curve of second kind. Then using the equation (3.2) we have
\[ \gamma = \frac{1}{k_1} \]
and considering the value of \( \gamma \) in the third equation of (3.2) we obtain
\[ \mu = -\frac{k'_1}{k_1^2 k_2} - \frac{\lambda k_1}{k_2} \]
where \( \lambda \) is a constant function. So, substituting the value of \( \mu, \gamma \) into the equation (3.1) we obtain the result.
Corollary 4.1. Let \( x : I \subset \mathbb{R} \rightarrow E^3 \) be a unit speed twisted curve in \( E^3 \). If \( x \) is a \( T \)-constant curve of second kind then the functions \( \lambda, \mu, \gamma \) satisfied the following equation
\[
\gamma^2 + \mu^2 = 2\lambda s + c \quad (4.3)
\]

**Proof.** Suppose that \( x \) is a \( T \)-constant curve of second kind. Then using the equation (3.2) we get
\[
k_1 = \frac{1}{\gamma}, \quad k_2 = -\frac{\mu'}{\gamma}.
\]
Then substituting these values into the third equation of (3.2) we have the following differential equation
\[
\gamma \gamma' + \mu \mu' = \lambda
\]
which is the solution of (4.3).

5. \( N \)-Constant Curves

**Definition 5.1.** Let \( x : I \subset \mathbb{R} \rightarrow E^n \) be a unit speed curve in \( E^n \). If \( \|x^N\| \) is constant then \( x \) is called a \( N \)-constant curve. For a \( N \)-constant curve \( x \) either \( \|x^N\| = 0 \) or \( \|x^N\| = \nu \) for some non-zero smooth function \( \nu \). Moreover, a \( N \)-constant curve \( x \) is called first kind if \( \|x^N\| = 0 \), otherwise second kind [6].

For a \( N \)-constant curve \( x \) the following equation satisfied
\[
\|x^N(s)\|^2 = \mu^2(s) + \gamma^2(s) = \omega \quad (5.1)
\]
where \( \omega \) is a constant function.

Considering the equation (3.1), (3.2) and (5.1) we obtain some results as follows.

**Lemma 5.1.** Let \( x : I \subset \mathbb{R} \rightarrow E^3 \) be a unit speed curve in \( E^3 \). Then \( x \) is a \( N \)-constant curve if and only if
\[
\begin{align*}
\lambda' &= 1 - \gamma k_1 \\
\mu' &= -\gamma k_2 \\
\gamma' &= \lambda k_1 + \mu k_2 \\
0 &= \gamma \gamma' + \mu \mu'
\end{align*}
\]
the above equation hold, where \( \lambda(s), \mu(s), \gamma(s) \) are differentiable functions.

**Proposition 5.1.** Let \( x : I \subset \mathbb{R} \rightarrow E^3 \) be a unit speed curve in \( E^3 \). Then \( x \) is a \( N \)-constant curve of first kind if and only if \( x(I) \) is an open portion of a straight line [3].

**Proof.** Let \( x \) is a \( N \)-constant curve in \( E^3 \), so the equation (5.1) holds. Moreover if \( x \) is a \( N \)-constant curve of first kind then using (5.1) we have \( \mu = \gamma = 0 \) which implies that \( k_1 = k_2 = 0 \). So \( x \) becomes a part of straight line.

**Theorem 5.1.** Let \( x : I \subset \mathbb{R} \rightarrow E^3 \) be a unit speed twisted curve in \( E^3 \). If \( x \) is a \( N \)-constant curve of second kind then the curve has the following parametrization
\[
x(s) = \left(-\frac{k_1'}{k_1 k_2}\right) N_2(s) + \frac{1}{k_1} B(s) \quad (5.3)
\]
or
\[
x(s) = (s + a) N_1(s) + c N_2(s) \quad (5.4)
\]
where \( a \) and \( c \) are real constants.
Proof. Suppose that \( x \) is a \( N \)-constant curve of second kind then substituting the second and third equation of (5.2) into the last equation of (5.2) we have
\[
\mu (-\gamma k_2) + \gamma (\lambda k_1 + \mu k_2) = 0
\]
\[
\gamma \lambda k_1 = 0
\]
Since \( k_1 \neq 0 \) we have two possibilities that \( \lambda = 0 \) or \( \gamma = 0 \). If \( \lambda = 0 \) then \( x \) is a \( T \)-constant curve and from the first and third equation of (5.2) \( x \) has the following parametrization
\[
x(s) = \left( -\frac{k_1'}{k_1^2 k_2} \right) N_2(s) + \frac{1}{k_1} B(s).
\]
(5.5)
If \( \gamma = 0 \) then using the equation (5.2) we obtain
\[
\begin{align*}
\lambda' &= 1 \\
\mu' &= 0
\end{align*}
\]
Then \( x \) satisfied the equation (5.4) which complete the proof.

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