

Bundle of Frames and Sprays for Fréchet Manifolds

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ABSTRACT

First we present a unified theory of connections on bundles necessary for the next studies. For a smooth manifold M , modeled on the Banach space \mathbb{B} , we define the bundle of linear frames LM and we endow it with a differentiable structure. Bundle of sprays FM , the pullback of LM via the tangent bundle $\pi : TM \rightarrow M$, is a natural bundle which provides us a rich environment to study the geometry of M . Afterward, despite of natural difficulties with Fréchet manifolds and even spaces, we generalize these results to a wide class of Fréchet manifolds, those which can be considered as projective limits of Banach manifolds. As an alternative approach we use pre-Finsler connections on FM and we show that our technique successfully solves ordinary differential equations on these manifolds. As some applications of our results we apply our method to enrich the geometry of two known Fréchet manifolds, i.e. jet of infinite sections and manifold of smooth maps, and we provide a suitable framework for further studies in these areas.

Keywords: Fréchet manifolds; frame bundle; spray bundle; projective limit; ordinary differential equations.

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1. Introduction

The study of infinite dimensional Fréchet manifolds was the subject of interest for many authors due to its interaction with mathematical physics and modern differential geometry [12, 17, 18, 19, 21]. The projective families of manifolds arise naturally in loop quantization, Gauge theories quantum gravity and 2D Yang-Mills theory where a compact space $\overline{\mathcal{A}/\mathcal{G}}$ extends the space of connections modulo gauge transformations [4].

Another area is String theory where Nag and Sullivan [17, 18] used the projective family of all finite sheeted compact unbranched coverings of a given closed Riemann surface of genus $g \geq 2$ which leads them to the concept of universal hyperbolic solenoid corresponds to the universal Teichmüller space \mathcal{T}_∞ . Also the space of connections via graphs in which one has a projective family of compact Hausdorff spaces labelled by an special partially ordered set called graphs [5].

To have an appropriate background in the study of infinite dimensional manifolds and bundles, we devote the first section give a unified framework for the study of connections on bundles, which plays an essential role in the next sections. (most of the results of this section are known but we could not find any comprehensive reference including different concepts of the connections altogether.

For a manifold M modeled on the Banach space \mathbb{E} the bundle of linear frames introduced as

$$LM = \bigcup_{x \in M} Lis(\mathbb{E}, T_x M).$$

Due to the fact that there is a one-to-one correspondence between connections on $\pi : TM \rightarrow M$ and principal connections on $p : LM \rightarrow M$ [25], the principal bundle LM , associated to the vector bundle TM , provides a rich environment for the study of geometric objects on M . Another interesting natural bundle is the **bundle of sprays** $FM := \pi^* LM$ i.e. the pullback of frame bundle along tangent bundle. This bundle provides us the opportunity to study the geometric objects on tangent and frame bundles simultaneously. More precisely we

will prove that FM is a principal bundle on M with the structure group $GL(\mathbb{E})$ associated to π^*TM . Introducing the concept of Pre-Finsler connections provides an alternative approach for the study of ordinary differential equations on Banach manifolds which is susceptible to be extended for projective limit Fréchet manifolds.

Taking one step further, we will study a wide class of Fréchet (non-Banach) manifolds. Various questions remains open in this area due to the natural difficulties in the study of these manifolds which is a reflection of intrinsic problems of their model spaces. The pathological structure of the general linear group and the lack of a general solvability theory of differential equations in non-Banach topological vector spaces are two problems of fundamental importance (cf. [12]).

We will prove that these problems will overcome if we restrict ourselves to those Fréchet manifolds which can be considered as projective limits of Banach manifolds. For example in this case for a projective limit manifolds $M = \varprojlim M^i$, $LM = \varprojlim LM^i$ and $FM = \varprojlim FM^i$ exists and they are generalized Fréchet principal bundles on $M = \varprojlim M^i$ and $TM = \varprojlim TM^i$ (see also [10, 11]).

Theorem 4.1 shows that projective limits technique successfully proposes an existence and uniqueness theorem for the solution of ordinary differential equations on this category of Fréchet manifolds. Our approach is supported with two application-examples i.e. manifolds of infinite jets and manifold of maps.

Through this paper all the maps and manifolds, for the sake of simplicity, are assumed to be smooth but (except for section 4) less degrees of differentiability may be assumed.

2. Connections

2.1. Connections on fibre bundles

Let $\pi : F \longrightarrow M$ be a fibre bundle with fibres N where M and N are Banach manifolds modeled on the Banach spaces \mathbb{B} and \mathbb{F} respectively. At every point $p \in F$ let $V\pi_p \subseteq T_pF$ be the vertical subspace, i.e. $V\pi_p = \ker T_p\pi$, and $V\pi = \bigcup_{p \in F} V\pi_p$ be the vertical subbundle. A connection on (F, π, M) , for abbreviation on π , is a smooth choice of $H\pi_p \subseteq T_pF$ for every $p \in E$, complementary to $V\pi_p$ with $TF = H\pi \oplus V\pi$. If π^h and π^v are the horizontal and vertical projections, then by a smooth choice of $H\pi$ we mean that $\pi^h(X)$ is smooth for every vector field $X \in \mathfrak{X}(F)$. Let (U, ϕ) be a chart for M such that $(\pi^{-1}(U), \Phi)$ is a local trivialization for π . In fact we consider a family of these trivializations for π which the domains cover M . Then

$$\Phi : \pi^{-1}(U) \longrightarrow \phi(U) \times N$$

and $T\Phi : \pi_F^{-1}(\pi^{-1}(U)) \longrightarrow \phi(U) \times N \times \mathbb{B} \times \mathbb{F}$ is the induced local trivialization for $\pi_F|_{\pi^{-1}(U)} : TF|_U \longrightarrow F|_U$. Let (V, Ψ) be another trivialization chart with $U \cap V \neq \emptyset$, then

$$\begin{aligned} T\Psi \circ T\Phi^{-1}(x, \xi, y, \eta) &= (\sigma_{\psi\phi}(x), G_{\psi\phi}(x, \xi), T\sigma_{\psi\phi}(x)y, \\ &T_1G_{\psi\phi}(x, \xi)y + T_2G_{\psi\phi}(x, \xi)\eta) \end{aligned} \quad (2.1)$$

where $\Psi \circ \Phi^{-1}(x, \xi) := (\sigma_{\psi\phi}(x), G_{\psi\phi}(x, \xi))$ and for $i = 1, 2$, T_i is the partial derivative with respect to the i -th variable. Then, there are smooth functions $\Gamma_\phi : \phi(U) \times N \longrightarrow L(\mathbb{B}, \mathbb{F})$ such that

$$\pi^v(x, \xi; y, \eta) := (x, \xi; 0, \eta + \Gamma_\phi(x, \xi)y).$$

In fact the differentiability of Γ_ϕ yields from the differentiability of the connection Γ . (Note that elements of $V\pi$ locally have the form $(x, \xi, 0, \eta)$ and π^v at every point is a linear projection.) Moreover we have

$$\pi^h(x, \xi; y, \eta) = (x, \xi; y, \eta) - \pi^v(x, \xi; y, \eta) = (x, \xi; y, -\Gamma_\phi(x, \xi)y).$$

With a little abuse of notation let $\{(\pi^{-1}(U), \Phi)\}$ be a family of local trivialization for $V\pi$. The compatibility condition for the local components yields from the fact that $T\Psi \circ T\Phi^{-1} \circ \pi^v|_U = \pi^v|_V \circ T\Psi \circ \Phi^{-1}$ and this holds if and only if for every $(x, \xi, y, \eta) \in \phi(U \cap V) \times N \times \mathbb{B} \times \mathbb{F}$;

$$\begin{aligned} T\Psi \circ T\Phi^{-1}(x, \xi, 0, \eta + \Gamma_\phi(x, \xi)y) &= \pi^v|_V((\sigma_{\psi\phi}(x), G_{\psi\phi}(x, \xi), \\ &T\sigma_{\psi\phi}(x)y, T_1G_{\psi\phi}(x, \xi)y + T_2G_{\psi\phi}(x, \xi)\eta) \\ \iff (\sigma_{\psi\phi}(x), G_{\psi\phi}(x, \xi), 0, 0 + T_2G_{\psi\phi}(x, \xi)[\eta + \Gamma_\phi(x, \xi)y]) &= (\sigma_{\psi\phi}(x), \\ G_{\psi\phi}(x, \xi), 0, T_1G_{\psi\phi}(x, \xi)y + T_2G_{\psi\phi}(x, \xi)\eta + \Gamma_\psi(\sigma_{\psi\phi}(x), G_{\psi\phi}(x, \xi)) &[T\sigma_{\psi\phi}(x)y]) \end{aligned}$$

iff the last components of both sides are equal i.e.

$$T_1 G_{\psi\phi}(x, \xi)y + \Gamma_{\psi}(\sigma_{\psi\phi}(x), G_{\psi\phi}(x, \xi))[T\sigma_{\psi\phi}(x)y] = T_2 G_{\psi\phi}(x, \xi)[\Gamma_{\phi}(x, \xi)y] \tag{2.2}$$

2.2. connections on vector bundles

Let $\pi : E \rightarrow M$ be a vector bundle with fibres isomorphic to the Banach space \mathbb{E} . Following the formalism of the previous part, for local trivializations (U, Φ) and (V, Ψ) with $U \cap V \neq \emptyset$, $\Psi \circ \Phi^{-1}(x, \xi) = (\sigma_{\psi\phi}(x), G_{\psi\phi}(x)\xi)$ where $G_{\psi\phi} : U \cap V \rightarrow GL(\mathbb{E})$ are smooth. (Here $GL(\mathbb{E})$ is the space of linear and continuous isomorphisms from \mathbb{E} to \mathbb{E}). Then the equation (2.1) takes the form

$$T(\Psi \circ \Phi^{-1})(x, \xi, y, \eta) = (\sigma_{\psi\phi}(x), G_{\psi\phi}(x)\xi, T\sigma_{\psi\phi}(x)y, G_{\psi\phi}(x)\eta + TG_{\psi\phi}(x)(y, \xi)). \tag{2.3}$$

and consequently the compatibility condition for the connection, say ∇ , is

$$G_{\psi\phi}(x)(\Gamma_{\phi}(x)[y, \xi]) = TG_{\psi\phi}(x)(y, \xi) + \Gamma_{\psi}(\sigma_{\psi\phi}(x))[T\sigma_{\psi\phi}(x)y, G_{\psi\phi}(x)\xi]. \tag{2.4}$$

The connection ∇ is linear if the local components are linear with respect to the second variable i.e. for every chart (U, ϕ) ,

$$\Gamma_{\phi} : \phi(U) \rightarrow \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{B}; \mathbb{E})).$$

Note that if M is finite dimensional and $E = TM$ we have the familiar notation

$$[\Gamma(x)(\xi, \eta)]^i = \sum \Gamma_{jk}^i(x)\xi^j\eta^k.$$

We define the connection map $\nabla : TE \rightarrow E$ locally given by

$$\nabla_{\phi}(x, \xi, y, \eta) := \Phi \circ \nabla \circ T\Phi^{-1}(x, \xi, y, \eta) = (x, \eta + \Gamma_{\phi}(x, \xi)y).$$

For every section ζ of π and every vector field X on M , the covariant derivative determined by ∇ is given by $\nabla_X \zeta := \nabla \circ T\zeta \circ X$ where $\zeta \in \Gamma(\pi)$ and $X \in \Gamma(\tau_M)$ [26].

But another familiar concept of connections is defined according to the covariant derivative properties. More precisely a connection is considered to be a map $\nabla : \Gamma(\pi) \times \Gamma(\tau_M) \rightarrow \Gamma(\pi)$ with the following properties; $\nabla_{X+Y}\zeta = \nabla_X\zeta + \nabla_Y\zeta$, $\nabla_X(\zeta + \zeta') = \nabla_X\zeta + \nabla_X\zeta'$ and $\nabla_f X\zeta = f\nabla_X\zeta$ where $\zeta, \zeta' \in \Gamma(\pi)$, $X, Y \in \Gamma(\tau_M)$ and $f \in C^\infty(M)$. Let (U, Φ) be a local trivialization for π . For $\zeta \in \Gamma(\pi)$ and $X \in \Gamma(\tau_M)$ suppose that $\bar{\zeta}_{\phi} := \text{proj}_2 \circ \Phi \circ \zeta$ and $\bar{X}_{\phi} := \text{proj}_2 \circ T\phi \circ X$ be the principal parts of ζ and X respectively.

Proposition 2.1. *Let ∇ be a connection on π . Then a unique covariant derivative can be defined which locally on (U, Φ) is given by*

$$(\nabla_X \zeta)|_U(\phi p) := (\nabla \circ T\zeta \circ X)|_U(\phi p) = d\bar{\zeta}_{\phi}(\phi p)\bar{X}_{\phi}(\phi p) + \omega_{\phi}(\phi p)[\bar{X}_{\phi}(\phi p), \bar{\zeta}_{\phi}(\phi p)].$$

Proof. See [14] and [22]. □

The converse is also true for the case that the total space of $\pi : E \rightarrow M$ is finite dimensional. We state the following theorem form [22].

Theorem 2.1. *Let $\pi : E \rightarrow M$ be a vector bundle with finite dimensional total space. Then for every covariant derivative on this bundle we can associate a linear connection.*

Here we state the definition of a metric from [13] which is also stated in [14]. For a vector bundle π we have the associated bundle $L_s^2(\pi) : L_s^2(E) \rightarrow M$ where $L_s^2(E)_p$ consists of the continuous symmetric bilinear maps from $\mathbb{E} \times \mathbb{E}$ to \mathbb{R} . Let $L_{\pi}^2(\mathbb{E})$ be the model of the fibres. It contains as an open subset $Ri(\mathbb{E})$ the positive definite forms, i.e. those forms which are $\geq \epsilon$ (Hilbert metric on \mathbb{E}), for some $\epsilon \geq 0$ [13].

Definition 2.1. A Riemannian metric on $\pi : E \rightarrow M$ is a differentiable section $g : M \rightarrow L_s^2(E)$ such that for every $p \in M$, $g(p)$ is positive definite. If we have a Riemannian metric g on $\tau_M : TM \rightarrow M$ then we call M a Riemannian manifold and we call g a Riemannian metric on M .

Theorem 2.2. *Let M be a manifold modeled on a self dual Banach space and ∇ a covariant derivative on M such that*

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for any $X, Y, Z \in \Gamma(\tau_M)$. Then a unique torsion free connection can be defined on M which is known as the Levi-Civita connection.

Proof. Let (U, ϕ) be a local chart on M . For any $(x, \xi, y, \eta) \in U \times \mathbb{B} \times \mathbb{B} \times \mathbb{B}$ the relation

$$g_U(\Gamma_U(x)[\xi, y], \eta) = \frac{1}{2}(dg_U(x).\xi(y, \eta) - dg_U(x).\eta(\xi, y) + dg_U(x).y(\xi, \eta))$$

defines the continuous and also smooth map $\Gamma_\phi : \phi(U) \longrightarrow L^2(\mathbb{B}, \mathbb{B})$ as the Christoffel symbols (for more details see [13], [14] or [22]). \square

2.3. connections on principal bundles

Let $\pi : P \longrightarrow M$ be a principal G -bundle with G a Banach Lie group modeled on the Banach space \mathbb{G} . Suppose that the right action of G on P is given by $r : P \times G \longrightarrow P; (p, g) \longmapsto p.g$. In fact locally on a chart (U, Φ) , $\Phi(r(\Phi^{-1}(x, h), g)) = (x, gh)$. For any $p \in P$ and any $g \in G$ consider the maps

$$r_p : G \longrightarrow P \quad \text{and} \quad r^g : P \longrightarrow P$$

given by $g \longmapsto p.g$ and $p \longmapsto p.g$ respectively.

In this case for every $p \in P$, $V\pi_p$ is isomorphic to \mathfrak{g} the Lie algebra of G . A principal connection on π is a smooth complementary, say $H\pi$, to $V\pi$ such that $H\pi$ (and consequently $V\pi$) is G -invariant i.e. $Tr^g H\pi = H\pi_{pg}$. In the other words for every $g \in G$,

$$Tr^g \circ \pi^v = \pi^v \circ Tr^g. \quad (2.5)$$

Hence, at every point, the vertical projection $\pi^v|_p : T_p P \longrightarrow V\pi_p$ can be considered as a \mathfrak{g} -valued 1-form $\Omega_p : T_p P \longrightarrow \mathfrak{g}$. Ω is called the connection form and

$$\Omega(X_p) = (T_e r_p)^{-1} \pi^v(X_p). \quad (2.6)$$

Clearly $\Omega(X^*) = X$ where X^* is the fundamental vector field associated to $X \in \mathfrak{g}$. In fact for every $X \in \mathfrak{g}$ the associated fundamental vector field (with values in $V\pi$) is

$$X^* : P \longrightarrow TP; \quad p \longmapsto T_e r_p X.$$

Moreover $(r^g)^* \Omega = Adg^{-1} \Omega$ since for every $p \in P$, $g \in G$ and $X_p \in T_p P$;

$$\begin{aligned} (r^g)^* \Omega(X_p) &= \Omega(T_p r^g X_p) = T_e(r_{pg}^{-1} \circ r_{pg}) \Omega(T_p r^g X_p) \\ &= T_{pg}(r_{ug})^{-1} \circ T_e r_{ug} \Omega(T_p r^g X_p) \\ &\stackrel{I}{=} T_{pg}(r_{pg})^{-1} \circ \pi^v(T_p r^g X_p) \\ &\stackrel{II}{=} T_{pg}(r_{pg})^{-1} \circ T_p r^g \circ \pi^v(X_p) \\ &= T_{pg}(r_{pg})^{-1} \circ T_p r^g \circ T_e r_p \circ (T_e r_p)^{-1} \pi^v(X_p) \\ &\stackrel{III}{=} T_{pg}(r_{pg})^{-1} \circ T_p r^g \circ T_e r_p \Omega(X_p) \\ &= Ad(g^{-1}) \Omega(X_p) \end{aligned}$$

In *II* we used (2.5) and for *I* and *III* the equality (2.6) is used.

To describe a principal connection locally let $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ be a family of trivializations for $\pi : P \longrightarrow M$. The physicist version of principal connection is based on the concept of local forms $\Omega_\alpha := s_\alpha^* \Omega : TU_\alpha \longrightarrow \mathfrak{g}$ where $\Phi_\alpha(s_\alpha(x)) = (x, e)$. Finally if $\{\Gamma_\alpha\}_{\alpha \in I}$ are the local components of the connection Ω , as described in section 1.1, then for every $(x, g, \xi) \in U_\alpha \times \mathbb{G} \times \mathbb{B}$;

$$\Gamma_\alpha(x, \xi)g = T_e \mu^g \Omega_\alpha(x, \xi)$$

in which $\mu^g : G \longrightarrow G; x \longmapsto x.g$. For a detailed study and further properties see [16] pp. 375-404.

3. Bundle of sprays for Banach manifolds

Let M be a smooth manifold modeled on the Banach space \mathbb{E} and $\mathcal{M} := \{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ be an atlas for M . For every $x \in M$ set

$$C_x = \{f : (-\epsilon, \epsilon) \longrightarrow M \text{ s.t. } \epsilon > 0, f(0) = x \text{ and } f \text{ is smooth}\}.$$

For $f, g \in C_x$ we define the equivalence relation; $f \sim_x g$ if and only if $f'(0) = g'(0)$. Then $TM = \bigcup_{x \in M} T_x M$ s.t. $T_x M = C_x / \sim_x$. The canonical vector bundle atlas for $\pi : TM \rightarrow M$ is $\mathcal{T} := \{(\pi^{-1}(U_\alpha), \Psi_\alpha)\}_{\alpha \in I}$ where

$$\Psi_\alpha : [f, x] \mapsto ((\psi \circ f)(0), (\psi \circ f)'(0)).$$

and $\pi : [f, x] \mapsto x$. Remind that the transition functions of TM are $\Psi_{\alpha\beta}(x) := d(\psi_\alpha \circ \psi_\beta^{-1})(x)$

3.1. Frame bundle

For any $x \in M$ define

$$L_x M := \{h : \mathbb{E} \rightarrow T_x M \text{ s.t. } h \text{ is a linear isomorphism}\}$$

Definition 3.1. The frame bundle of M is the (disjoint) union $LM := \bigcup_{x \in M} L_x M$.

Dodson and Galanis in [7] proved that for the case of second order tangent bundle the corresponding frame bundle is a principal bundle associated to the vector bundle $T^2 M$. We follow the techniques of [7] and [25].

Consider the canonical projection $p : LM \rightarrow M$ given by $p(h) = x$ whenever $h \in L_x M$ (or h is a linear frame at x).

Proposition 3.1. (LM, p, M) is a principal fibre bundle with the structure group $GL(\mathbb{E})$.

Proof. The natural family of local trivializations for LM is $\mathcal{L} := \{(p^{-1}(U_\alpha), \phi_\alpha)\}_{\alpha \in I}$ s.t.

$$\phi_\alpha : p^{-1}(U_\alpha) \rightarrow \phi_\alpha(U_\alpha) \times \mathbb{E}; h \mapsto (\psi_\alpha(p(h)), (pr_2 \circ \Psi_\alpha(x)) \circ h).$$

Clearly $\{p^{-1}(U_\alpha)\}_{\alpha \in I}$ covers LM . For any $\alpha \in I$, ϕ_α is well defined. Moreover ϕ_α is injective since $\psi_\alpha(p(h) = \psi_\alpha(p(h'))$ yields that h and h' are frames at x . Since $\Psi_\alpha(x) := \Psi_\alpha|_{\pi^{-1}(x)}$ is a linear isomorphism we see that $h = h'$. For every $(x, g) \in \psi_\alpha(U_\alpha) \times \mathbb{E}$ we have $\phi_\alpha(x, \Psi_\alpha^{-1}(x) \circ g) = (x, g)$ which shows that ϕ_α is surjective. Furthermore for any $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$, $\phi_\alpha(U_\alpha \cap U_\beta) = \psi_\alpha(U_\alpha \cap U_\beta) \times GL(\mathbb{E})$ is open in the Banach space $\mathbb{E} \times L(\mathbb{E}, \mathbb{E})$ and

$$\phi_\alpha \circ \phi_\beta^{-1}(x, g) = (\psi_\alpha \circ \psi_\beta^{-1}(x), \Psi_{\alpha\beta}(x) \circ g)$$

is a smooth diffeomorphism. As a consequence, LM admits a smooth manifold structure. Moreover $\{(U_\alpha, p_\alpha)\}_{\alpha \in I}$ forms a family of trivializations which makes p a $GL(\mathbb{E})$ principal bundle. The right action of $GL(\mathbb{E})$ on LM is given by $\phi_\alpha(\phi_\alpha^{-1}(x, h).g) := (x, g \circ h) = (x, gh)$ and the smooth transition functions are given by

$$\phi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(\mathbb{E}); x \mapsto \phi_{\alpha,x} \circ \phi_{\beta,x}^{-1}.$$

□

Define the action $\mathcal{R}_{LM} : (LM \times \mathbb{E}) \times GL(\mathbb{E}) \rightarrow LM \times \mathbb{E}; ((h, v).g) \mapsto (h \circ g, g^{-1}v)$.

Proposition 3.2. The quotient space $(LM \times \mathbb{E})/GL(\mathbb{E})$ is isomorphic with the tangent bundle $\pi : TM \rightarrow M$.

Proof. First we prove that

$$\begin{aligned} \tilde{\pi} : (LM \times \mathbb{E})/GL(\mathbb{E}) &\rightarrow M \\ [(h, v)] &\mapsto p(h) \end{aligned}$$

is a vector bundle and then we conclude that it is isomorphic to π . ($[(h, v)]$ denotes the equivalence class containing (h, v) with respect to the equivalence relation induced by \mathcal{R}_{LM} .)

Clearly $\tilde{\pi}$ is well defined. According to the atlas $\mathcal{M} = \{U_\alpha, \psi_\alpha\}_{\alpha \in I}$ define the

$$\begin{aligned} \tilde{\Psi}_\alpha : \tilde{\pi}^{-1}(U_\alpha) &\rightarrow \psi_\alpha(U_\alpha) \times \mathbb{E} \\ [h, v] &\mapsto (p(h), pr_2 \circ \phi_\alpha(hv)) \end{aligned}$$

$\tilde{\Psi}_\alpha$ is well defined since for another representative of the class $[h, v]$, like $(h, v).g$ for some $g \in GL(\mathbb{E})$, we see that

$$\tilde{\Psi}_\alpha((h, v).g) = (p(h \circ g), pr_2 \circ \phi_\alpha \circ h \circ g(g^{-1}v)) = \tilde{\Psi}_\alpha((h, v))$$

Let $[h, v], [h', v'] \in \tilde{\pi}^{-1}(U_\alpha)$ such that $\tilde{\Psi}_\alpha([h, v]) = \tilde{\Psi}_\alpha([h', v'])$. Then $p(h) = p(h') = x$ i.e. h and h' are frames at x . Moreover $pr_2 \circ \phi_\alpha(h(v)) = \phi_{\alpha,x}(h(v)) = \phi_{\alpha,x}(h'(v')) = pr_2 \circ \phi_\alpha(h'(v'))$ which implies that $h(v) = h'(v')$. Then via the linear isomorphism $g := h^{-1} \circ h'$ the pairs $[h, v], [h', v']$ are equivalent i.e. $\tilde{\Psi}_\alpha$ is injective. $\tilde{\Psi}_\alpha$ is also surjective since for $(x, v) \in \phi_\alpha(U_\alpha) \times \mathbb{E}$, $\tilde{\Psi}_\alpha([\phi_{\alpha,x}^{-1}, v]) = (x, v)$. Furthermore for $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$ we see that $\tilde{\Psi}_{\alpha\beta} := \tilde{\Psi}_\alpha \circ \tilde{\Psi}_\beta^{-1}|_{x \times \mathbb{E}} = \Psi_{\alpha\beta}(x)$.

Define the bundle morphism

$$\begin{aligned} \Xi_{LM} : ((LM \times \mathbb{E})/GL(\mathbb{E}), \tilde{\pi}, M) &\longrightarrow (TM, \pi, M) \\ [h, v] &\longmapsto (p(h), h(v)) \end{aligned}$$

It is easily seen that Ξ_{LM} is well defined. Let $\Xi_{LM}([h, v]) = \Xi_{LM}([h', v'])$ then we conclude that h and h' are frames at the same point i.e. $x := p(h) = p(h')$. It is easy to see that $[h, v]$ and $[h', v']$ are equal via the isomorphism $g := h^{-1} \circ h'$ which means that Ξ_{LM} is injective.

Ξ_{LM} is surjective. To see that, let $\Psi_\alpha([f, x]) = (x, v) \in TM$ be arbitrary. Then

$$[f, x] = \Xi_{LM}([\Psi_\alpha^{-1}(x), \Psi_\alpha(x)([f, x])]).$$

In fact for a chart (U_α, ψ_α) of M if we consider the induced trivializations $(\pi^{-1}(U_\alpha), \Psi_\alpha)$ and $(\tilde{\pi}^{-1}(U_\alpha), \tilde{\Psi}_\alpha)$ for π and $\tilde{\pi}$ respectively, then Ξ_{LM} maps $\pi^{-1}(U_\alpha)$ diffeomorphically to $\tilde{\pi}^{-1}(U_\alpha)$. \square

3.2. Connections and Parallelism on frame bundle

Let Ω be a principal connection on (LM, p, M) and $c : (\epsilon, \epsilon) \rightarrow M$ a smooth curve. We say that $C : (\epsilon, \epsilon) \rightarrow LM$ covers c if $p \circ C(t) = c(t)$ for every $t \in (\epsilon, \epsilon)$.

Definition 3.2. The curve $C(t)$ is called **horizontal** with respect to Ω (or horizontal lift of c) if $C'(t)$ lies in the horizontal subspace $H_p \subseteq TLM$ or equivalently $p^v(C'(t)) = 0$.

Here $TLM = V_p \oplus H_p$ and p^v and p^h are the vertical and the horizontal projections respectively.

Let (U_α, ψ_α) be a local chart around $c(0) = x$ and (U_α, Ψ_α) and (U_α, ϕ_α) the corresponding local trivializations for TM and LM respectively. Then $\phi_\alpha \circ C(t) := (c(t), z(t))$ and $T\phi_\alpha \circ C'(t) := (c(t), z(t), c'(t), z'(t))$ and consequently C is horizontal if and only if (locally)

$$p^v(C'(t)) = z'(t) + \Gamma_\alpha((c(t), z(t)))c'(t) = 0$$

Theorem 3.1. For a curve c in M and $h_0 \in L_{x_0}M$, there exists a unique (global) horizontal lift C in LM s.t. $C(0) = h_0$.

Proof. The proof is a direct consequence of the existence and uniqueness theorem for ordinary differential equations on Banach spaces (see e.g. [14]). \square

3.3. Bundle of sprays

Let M be a Banach manifold modeled on \mathbb{B} with (TM, π, M) and (LM, p, M) the tangent and frame bundles of M respectively. Following the definition of spray bundle given by Antonelli etc. ([3]) for finite dimensional manifolds we state this definition.

Definition 3.3. The pullback of LM via π , $FM := \pi^*LM$, is called the **spray bundle** and is denoted by (FM, ρ, TM) .

We will see that this natural bundle provides a rich environment to study geometric objects in tangent and frame bundles simultaneously. In fact FM is the space of all pairs $\xi = ([f, x], h)$ where $[f, x]$ is a tangent vector at $x = \pi([f, x])$ and h is a frame at x . FM is a Banach manifold modeled on $(\mathbb{E} \times \mathbb{E}) \times GL(\mathbb{E})$ with the canonical atlas $\mathcal{F} = \{(\rho^{-1}(\pi^{-1}(U_\alpha)), \vartheta_\alpha)\}_{\alpha \in I}$ and for any $\alpha \in I$;

$$\begin{aligned} \vartheta_\alpha : \rho^{-1}(\pi^{-1}(U_\alpha)) &\longrightarrow \mathbb{E} \times \mathbb{E} \times GL(\mathbb{E}) \\ ([f, x], h) &\longmapsto (\Psi_\alpha([f, x]), \phi_\alpha(h)) \end{aligned}$$

We can prove the following proposition just like proposition 3.1.

Proposition 3.3. FM is a principal bundle over TM with the structure group $Gl(\mathbb{E})$.

For every $\xi \in FM$, the vertical subspace is defined to be $V_{\rho\xi} := \{X \in T_{\xi}FM \mid T\rho(X) = 0\}$. Let Γ be a principal connection on ρ i.e. a smooth complementary $H\rho$ to $V\rho$ s.t. $H\rho$ is $GL(\mathbb{E})$ -invariant.

Definition 3.4. A **Pre-Finsler connection** $F\Gamma$ on (FM, ρ, TM) is a pair (Γ, D) where Γ is a principal connection on ρ and D is a connection (possibly nonlinear) on $\pi : TM \rightarrow M$.

For every $([f, x], h) = \xi \in FM, T_{\xi}\rho : T_{\xi}FM \rightarrow T_{[f,x]}TM$ is a surjective linear map with the kernel $V_{\xi}\rho$. Hence $T_{\xi}\rho : H_{\xi}\rho \rightarrow T_{[f,x]}TM$ is a linear isomorphism. The inverse of $T_{\xi}\rho$ is called horizontal lift with respect to Γ and we denote it by $\mathbf{L}_{\rho,\xi}$. In a same way one can define horizontal lift w.r.t. π by the map $\mathbf{L}_{\pi,[f,x]} : T_xM \rightarrow H_{[f,x]}\pi$. As a result at the presence of pre-Finsler connection (Γ, D) , there are two different types of Horizontal subspaces, the **h-horizontal** subspace denoted by $H_{\xi}^{hor}\rho := \mathbf{L}_{\rho,\xi}(H_{[f,x]}\pi)$ and the **v-horizontal** subspace $H_{\xi}^{ver}\rho := \mathbf{L}_{\rho,\xi}(V_{[f,x]}\pi)$.

Let $(U_{\alpha}, \psi_{\alpha})$ be a local chart around $x \in M$. Suppose that $\Psi_{\alpha}([f, x]) = (x, v) \in \psi_{\alpha}(U_{\alpha}) \times \mathbb{E}$ and consider a vector field X on M which is locally given by $\Psi_{\alpha} \circ X(x) = (x, \bar{X}_{\alpha}(x))$. Then

$$T\Psi_{\alpha} \circ \mathbf{L}_{\pi,[f,x]}(x, \bar{X}_{\alpha}(x)) = (x, v, \bar{X}_{\alpha}(x), -\omega_{\alpha}(x)[\bar{X}_{\alpha}(x), v])$$

In fact suppose that $T\psi_{\alpha} \circ \mathbf{L}_{\pi,[f,x]}(x, \bar{X}_{\alpha}(x)) = (a, b, c, d)$ then **i.** (a, b, c, d) belongs to $T_{[f,x]}TM$ i.e. $(a, b) = (x, v)$. **ii.** $T_{[f,x]}\pi(x, v, c, d) = X(x)$ i.e. $c = \bar{X}_{\alpha}(x)$. **iii.** $(x, v, \bar{X}_{\alpha}(x), d)$ is horizontal which means that $d = -\omega_{\alpha}(x, \bar{X}_{\alpha}(x))v$. ($\{\omega_{\alpha}\}_{\alpha \in I}$ and $\{\Gamma_{\alpha}\}_{\alpha \in I}$ denote the local components of D and Γ respectively.)

The local form of $\mathbf{L}_{\rho,\xi}$ can be described in the same way. Moreover, here we have the natural right action of $GL(\mathbb{E})$ on FM given by

$$([f, x], h); g \mapsto ([f, x], h \circ g).$$

If we define

$$\mathcal{R}_{FM} : (FM \times \mathbb{E}) \times GL(\mathbb{E}) \rightarrow FM; ([f, x], h), v; g \mapsto ([f, x], h \circ g), g^{-1}v).$$

then we have the following theorem.

Theorem 3.2. The quotient space $FM \times \mathbb{E}/GL(\mathbb{E})$ is isomorphic to the vector bundle (π^*, TM, π^*TM) . (π^*, TM, π^*TM) is the pullback of the tangent bundle via itself.

Proof. We just introduce the projection and trivializations and the details can be proved like theorem 3.2. The projection is

$$\begin{aligned} \tilde{\rho} : (FM \times \mathbb{E})/GL(\mathbb{E}) &\rightarrow TM \\ [[f, x], h), v] &\mapsto [f, x] \end{aligned}$$

and the corresponding local trivialization (chart) induced from the chart $(U_{\alpha}, \psi_{\alpha})$ of M is

$$\begin{aligned} \tilde{\vartheta}_{\alpha} : \tilde{\rho}^{-1}(\pi^{-1}(U_{\alpha})) &\rightarrow (\psi_{\alpha}(U_{\alpha}) \times \mathbb{E}) \times \mathbb{E} \\ [[f, x], h), v] &\mapsto (\Psi_{\alpha}([f, x]), pr_2 \circ \phi_{\alpha} \circ h(v)) \end{aligned}$$

The transition functions of the bundle $\tilde{\rho}$ are those of $\pi : TM \rightarrow M$. Finally the map

$$\begin{aligned} \Xi_{FM} : (FM \times \mathbb{E})/GL(\mathbb{E}) &\rightarrow \pi^*TM \\ [[f, x], h), v] &\mapsto ([f, x], h(v)) \end{aligned}$$

is the desired vector bundle isomorphism. □

3.4. Pre-Finsler Parallelism

Let (Γ, D) be a pre-Finsler connection on (FM, ρ, TM) and $\bar{C} : (-\epsilon, \epsilon) \rightarrow FM$ and $C : (-\epsilon, \epsilon) \rightarrow TM$ be smooth curves with $\rho \circ \bar{C} = C$. Then we say that \bar{C} covers C . \bar{C} is called a Γ -horizontal lift of C if $\bar{C}'(t)$ is Γ -horizontal for all t . Equivalently the vertical projection w.r.t. Γ , $\rho^v : TFM \rightarrow VFM$, vanishes $\bar{C}'(t)$, i.e. $\rho^v(\bar{C}'(t)) = 0$. Consider a local chart $(U_{\alpha}, \psi_{\alpha})$ of M around $\pi \circ C(0) = x$. Suppose that $(\pi^{-1}(U_{\alpha}), \Psi_{\alpha})$ and $(\rho^{-1}(\pi^{-1}(U_{\alpha})), \vartheta_{\alpha})$ be the corresponding induced trivializations for TM and FM respectively. If $\vartheta_{\alpha} \circ \bar{C}(t) =$

$(x(t), y(t), z(t))$ then $\Psi_\alpha \circ C(t) = (x(t), y(t))$ and $\psi_\alpha \circ c(t) := \psi_\alpha \circ \pi \circ C(t) = x(t)$. Then \bar{C} is Γ -horizontal lift of C if for every $\alpha \in I$

$$\begin{aligned} T\vartheta_\alpha \circ \rho^v \circ \bar{C}'(t) = \\ (x(t), y(t); z(t), 0, 0; z'(t) + \Gamma_\alpha(x(t), y(t); z(t))[x'(t), y'(t)]) = 0. \end{aligned} \quad (3.1)$$

Here $\Gamma_\alpha : \Psi_\alpha(U_\alpha) \times GL(\mathbb{E}) \rightarrow L(\mathbb{E} \times \mathbb{E}, GL(\mathbb{E}))$, $\alpha \in I$, are the local components of the principal connection Γ on ρ .

The curve C is called the D -horizontal lift of $c := \pi \circ C$ if $\pi^v(C'(t)) = 0$ or locally

$$\begin{aligned} T\Psi_\alpha \circ \pi^v(x(t), y(t); x(t)', y(t)') = \\ (x(t), y(t); 0, y'(t) + \omega_\alpha(x(t), x'(t))y(t)) = 0; \quad \alpha \in I. \end{aligned} \quad (3.2)$$

Definition 3.5. Let $\bar{C} : (-\epsilon, \epsilon) \rightarrow FM$ and $c : (-\epsilon, \epsilon) \rightarrow M$ be smooth curves such that $\rho \circ \pi \circ \bar{C} = c$. Then the curve \bar{C} is called (Γ, D) -horizontal lift of c if there exists a curve $C : (-\epsilon, \epsilon) \rightarrow TM$ s.t. C is D -horizontal lift of c and \bar{C} is the Γ -horizontal lift of C .

Proposition 3.4. Let $c : (-\epsilon, \epsilon) \rightarrow M$ be a smooth curve with $c(0) = x_0$ and (Γ, D) be a pre-Finsler connection on M . Then for every $\xi_0 = (x_0, y_0, z_0) \in \psi_\alpha(U_\alpha) \times \mathbb{E} \times GL(\mathbb{E})$ locally there exists a unique (Γ, D) -horizontal lift of c to FM , say \bar{C} , s.t. $\bar{C}(0) = \xi_0$.

Proof. Let $C : (-\epsilon, \epsilon) \rightarrow TM$ be a curve above c which locally on a chart is given by $C(t) = (x(t), y(t))$. Then C is a D -horizontal lift of c if and only if

$$y(t)' + \omega_\alpha(x(t), x'(t))y(t) = 0.$$

Hence by the existence and uniqueness theorem for ordinary differential equations on Banach spaces, for (x_0, y_0) there exists a unique horizontal lift C for c s.t. $C(0) = (x_0, y_0)$.

\bar{C} is Γ -horizontal lift of C if locally the equation

$$z'(t) + \Gamma_\alpha(x(t), y(t); z(t))[x'(t), y'(t)] = 0.$$

Again according to the existence and uniqueness of ordinary differential equations on Banach spaces (see for example [14]), for C there exists a unique Γ -horizontal lift say \bar{c} s.t. $\bar{C}(0) = \xi_0$. One can merge the ordinary differential equations of both steps to get the following system of differential equations which determines the (Γ, D) -horizontal lift of a curve c

$$\begin{aligned} y(t)' + \omega_\alpha(x(t), x'(t))y(t) = 0 \\ z'(t) + \Gamma_\alpha(x(t), y(t); z(t))[x'(t), y'(t)] = 0; \quad \alpha \in I. \end{aligned}$$

□

4. The Fréchet case

In the sequel we introduce our notations about a special class of Fréchet manifolds which are obtained as projective limits of Banach manifolds. Suppose that $\{M^i, \mathcal{M}^{ji}\}_{i,j \in \mathbb{N}}$ be a projective system of Banach manifolds. This last means that for any $i \in \mathbb{N}$, M^i is a Banach manifold modeled on the Banach space \mathbb{E}^i and for $j \geq i$, $\mathcal{M}^{ji} : M^j \rightarrow M^i$ is a differentiable map with $\mathcal{M}^{ii} = id_{M^i}$ and $\mathcal{M}^{ji} \circ \mathcal{M}^{kj} = \mathcal{M}^{ki}$ for all $k \geq j \geq i$. Then the limit $M = \varprojlim M^i$ is a subset of $\prod_{i=1}^{\infty} M^i$ consisting all threads $(m_i)_{i \in \mathbb{N}} \in \prod_{i=1}^{\infty} M^i$ such that $\mathcal{M}^{ji}(m_j) = m_i$ for all $j \geq i$. Moreover we need that the Banach spaces $\{\mathbb{E}^i, \lambda^{ji}\}_{i \in \mathbb{N}}$ also forms a projective system of Banach spaces. For $j \geq i$, $\lambda^{ji} : \mathbb{E}^j \rightarrow \mathbb{E}^i$ are continuous linear maps and $\varprojlim \mathbb{E}^i := \mathbb{F}$ (for more details see [24, 23]).

Furthermore let for every $x = (x_i)_{i \in \mathbb{N}} \in M$ there exists a projective system of local charts $\{(U^i, \psi^i)\}_{i \in \mathbb{N}}$ such that $x^i \in U^i$ and $U = \varprojlim U^i$ is open in M . In fact $M = \varprojlim M^i \subseteq \prod_{i \in \mathbb{N}} M^i$ consists of those elements $(x^i)_{i \in \mathbb{N}}$ s.t. $\mathcal{M}^{ji}(x^j) = x^i$ for every $j \geq i$.

Let $\{M^i, \mathcal{M}^{ji}\}_{i,j \in \mathbb{N}}$ and $\{N^i, \mathcal{N}^{ji}\}_{i,j \in \mathbb{N}}$ be projective families of manifolds. A family of maps $\{\mu^i : M^i \rightarrow N^i\}_{i \in \mathbb{N}}$ is a projective system of maps if $\mathcal{N}^{ji} \circ \mu^j = \mu^i \circ \mathcal{M}^{ji}$ for $i, j \in \mathbb{N}$ with $j \geq i$ [10, 24]. We alternatively use the notations $\mu = \varprojlim \mu^i$, μ and $(\mu^i)_{i \in \mathbb{N}}$ when there is no confusion.

The vector bundle structure of TM for a Fréchet manifold M links to pathological structure of general linear group $GL(\mathbb{F})$ and this causes troubles. It is shown [10] that by considering the generalized topological Lie group

$$\mathcal{H}_0(\mathbb{F}) := \varprojlim \mathcal{H}_0^i(\mathbb{E}^i) = \{(l^i)_{i \in \mathbb{N}} \in \prod_{i \in I} GL(\mathbb{E}^i) : \varprojlim l^i \text{ exists}\}$$

rather than $GL(\mathbb{F})$ this obstacle can be solved. Moreover as we will see in the rest of this paper, the problems related to the lack of a general solvability for differential equations on Fréchet manifolds will overcome with our technique.

Galanis in [10] proved that if $\{M^i, \mathcal{M}^{ji}\}_{i,j \in \mathbb{N}}$ is a projective system of manifolds then $\{TM^i, \mathcal{T}^{ji}\}_{i,j \in \mathbb{N}}$ also form a projective system of manifolds. In this case TM is isomorphic (set-theoretically) to $\varprojlim TM^i$ and TM admits a generalized vector bundle structure with the structure group $\mathcal{H}_0(\mathbb{F})$.

For any $i \in \mathbb{N}$ and $x^i \in M^i$ define

$$LM^i := \bigcup_{x^i \in M^i} \{(h^k)_{1 \leq k \leq i} \in \prod_{k=1}^i GL(\mathbb{E}^k, T_{\mathcal{M}^{ik}(x^i)} M^k) \text{ s.t.} \\ \mathcal{T}^{jk} \circ h^j = h^k \circ \lambda^{jk} \text{ for } 1 \leq k \leq j \leq i\}$$

and

$$\mathcal{H}_0^i(\mathbb{E}^i) := \{(g^k)_{1 \leq k \leq i} \in \prod_{k=1}^i GL(\mathbb{E}^k) : \lambda^{jk} \circ g^j = g^k \circ \lambda^{jk} \text{ for all, } 1 \leq k \leq j \leq i\}$$

with the connecting morphisms $\mathcal{H}^{ji} : \mathcal{H}_0^j(\mathbb{E}^j) \rightarrow \mathcal{H}_0^i(\mathbb{E}^i); (g^1, g^2, \dots, g^j) \mapsto (g^1, g^2, \dots, g^i)$ for $j \geq i$.

Lemma 4.1. For any $i \in \mathbb{N}$, LM^i is a principal bundle on M^i with the structure group $\mathcal{H}_0^i(\mathbb{E}^i)$ and its right action is given by

$$\mathcal{R}^i : LM^i \times \mathcal{H}_0^i(\mathbb{E}^i) \rightarrow LM^i; ((h^k)_{1 \leq k \leq i}, (g^k)_{1 \leq k \leq i}) \mapsto (h^k \circ g^k)_{1 \leq k \leq i}.$$

Proof. Let $x^i \in M^i$. Since the projection $\mathcal{M}^i : \varprojlim M^i \rightarrow M^i; (x^i)_{i \in \mathbb{N}} \mapsto x^i$ is surjective, then we can consider $(x^i)_{i \in \mathbb{N}}$ s.t. $\mathcal{M}^i((x^i)_{i \in \mathbb{N}}) = x^i$. For this element there exists a projective family of charts $\{(U_\alpha^i, \psi_\alpha^i)\}_{i \in \mathbb{N}}$ s.t. for every $i \in \mathbb{N}$, $x^i \in U_\alpha^i$. Consider the induced projective family of trivialization $\{(\pi_i^{-1}(U_\alpha^i), \Psi_\alpha^i)\}_{i \in \mathbb{N}}$ for $TM = \varprojlim TM^i$. For $\alpha \in I$ define the family of trivializations

$$\Phi_\alpha^i : p_i^{-1}(U_\alpha^i) \rightarrow \psi_\alpha^i(U_\alpha^i) \times \mathcal{H}_0^i(\mathbb{E}^i) \\ (h^1, \dots, h^i) \mapsto (p^i(h^1, \dots, h^i), (\Psi_\alpha^1(x^1) \circ h^1, \dots, \Psi_\alpha^i(x^i) \circ h^i))$$

Clearly $\{p_i^{-1}(U_\alpha^i)\}_{\alpha \in I}$ covers LM^i . Φ_α^i is injective since if $\Phi_\alpha^i(h^1, \dots, h^i) = \Phi_\alpha^i(\bar{h}^1, \dots, \bar{h}^i)$ then $p_i(h^1, \dots, h^i) = p_i(\bar{h}^1, \dots, \bar{h}^i) := x^i$ yields that h^k and \bar{h}^k are frames at $x^k := \mathcal{M}^{ik}(x^i)$ for any $1 \leq k \leq i$. Moreover $\Psi_\alpha^k(x^k) \circ h^k = \Psi_\alpha^k(x^k) \circ \bar{h}^k$ which shows that $(h^1, \dots, h^i) = (\bar{h}^1, \dots, \bar{h}^i)$.

Furthermore Φ_α^i is surjective since for every $x^i \in U_\alpha^i$ and $(g^1, \dots, g^i) \in \mathcal{H}_0^i(\mathbb{E}^i)$,

$$\Phi_\alpha^i(\Psi_\alpha^1(x^1)^{-1} \circ g^1, \dots, \Psi_\alpha^i(x^i)^{-1} \circ g^i) = (x^i; g^1, \dots, g^i).$$

Let $\alpha, \beta \in I$ and $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$. Then

$$\Phi_\alpha^i \circ \Phi_\beta^{i-1} : \psi_\beta^i(U_{\alpha\beta}^i) \times \mathcal{H}_0^i(\mathbb{E}^i) \rightarrow \psi_\alpha^i(U_{\alpha\beta}^i) \times \mathcal{H}_0^i(\mathbb{E}^i) \\ (x^i, (g^1, \dots, g^i)) \mapsto (\psi_\alpha^i \circ \psi_\beta^{i-1}(x^i), (\Psi_{\alpha\beta}^1(x^1), \dots, \Psi_{\alpha\beta}^i(x^i)))$$

where $\Psi_{\alpha\beta}^k : U_{\alpha\beta}^k \rightarrow GL(\mathbb{E}^k)$ are transition functions on TM^k for $1 \leq k \leq i$. Note that $\mathcal{H}_0^i(\mathbb{E}^i)$ is an open subset of the Banach space $\prod_{k=1}^i L(\mathbb{E}^k, \mathbb{E}^k)$. Then according to gluing lemma ([6], 5.2.4), LM^i is a principal bundle on M^i with the structure group $\mathcal{H}_0^i(\mathbb{E}^i)$. It is easily seen that \mathcal{R}^i is the natural induces action of $\mathcal{H}_0^i(\mathbb{E}^i)$ on LM^i (see also [8] and [23]). \square

Remarks 1. One can use the proof of the previous lemma to show that, analogously, for every $i \in \mathbb{N}$;

$$FM^i := \bigcup_{x^i \in M^i} \{([f^i, x^i]^i; (h^1, \dots, h^i)) \text{ s.t. } [f^i, x^i]^i \in T_{x^i} M^i, \\ h^k \in Lis(\mathbb{E}^k, T_{x^k} M^k) \text{ and } \mathcal{T}^{jk} \circ h^j = h^k \circ \lambda^{jk} \text{ for } 1 \leq k \leq j \leq i\}$$

is a principal bundle on TM^i with the structure group $\mathcal{H}_0^i(\mathbb{E}^i)$. For $i \in \mathbb{N}$ let $\rho^i : FM^i \rightarrow TM^i$; $([f^i, x^i]^i; (h^1, \dots, h^i)) \mapsto [f^i, x^i]^i$ be the natural projection then, the corresponding family of trivializations for FM^i is given by

$$\begin{aligned} \Phi_{\alpha, FM^i}^i : \rho_i^{-1}(\pi_i^{-1}(U_\alpha^i)) &\longrightarrow \psi_\alpha^i(U_\alpha^i) \times \mathbb{E}^i \times \mathcal{H}_0^i(\mathbb{E}^i) \\ ([f^i, x^i]^i; (h^1, \dots, h^i)) &\longmapsto \left(\psi_\alpha^i(x^i), (\psi_\alpha^i \circ f^i)'(0); (\Psi_\alpha^1(x^1) \circ h^1, \dots, \Psi_\alpha^i(x^i) \circ h^i) \right) \end{aligned}$$

and the induced right action is given by

$$\begin{aligned} \mathcal{R}_{i, FM} : FM^i \times \mathcal{H}_0^i(\mathbb{E}^i) &\longrightarrow FM^i \\ \left(([f^i, x^i]^i; (h^1, \dots, h^i)), (g^1, \dots, g^i) \right) &\longmapsto \left(([f^i, x^i]^i; (h^1 \circ g^1, \dots, h^i \circ g^i)) \right) \end{aligned}$$

For $j \geq i$, define the connecting morphism

$$\mathcal{L}^{ji} : LM^j \longrightarrow LM^i; (h^1, \dots, h^j) \longmapsto (h^1, \dots, h^i)$$

and

$$\mathcal{H}^{ji} : \mathcal{H}_0^j(\mathbb{E}^j) \longrightarrow \mathcal{H}_0^i(\mathbb{E}^i); (g^1, \dots, g^j) \longmapsto (g^1, \dots, g^i)$$

Proposition 4.1. *The following statements hold true.*

- 1.) $\{LM^i, \mathcal{H}_0^i(\mathbb{E}^i), M^i, \rho^i\}_{i \in \mathbb{N}}$ forms a projective system of principal bundles with the corresponding family of connecting morphism $\{\mathcal{L}^{ji}\}_{i, j \in \mathbb{N}}$.
2. $\{FM^i, \mathcal{H}_0^i(\mathbb{E}^i), TM^i, \rho^i\}_{i \in \mathbb{N}}$ is a projective system of principal bundles over $\{TM^i\}_{i \in \mathbb{N}}$ with the connecting morphisms

$$\begin{aligned} \mathcal{F}^{ji} : FM^j &\longrightarrow FM^i \\ ([f^j, x^j]^j, (h^1, h^2, \dots, h^j)) &\longmapsto ([f^i, x^i]^i, (h^1, h^2, \dots, h^i)) \end{aligned}$$

where $f^i := \mathcal{M}^{ji} \circ f^j$ for any pair of integers i, j with $j \geq i$.

3. $\varprojlim LM^i$ is a Fréchet principal bundle, over $M = \varprojlim M^i$, with the generalized Lie group $\mathcal{H}_0(\mathbb{F})$ as its the structure group.
4. $\varprojlim FM^i$ is a $\mathcal{H}_0(\mathbb{F})$ -principal bundle over $TM = \varprojlim TM^i$.

Proof. **1,2.** We prove 2. and 1. can be proved analogously. For any $x = (x^i)_{i \in \mathbb{N}} \in M$, consider a projective limit chart $\{(U_\alpha^i, \psi_\alpha^i)\}$ and the induced trivializations $\{(\Phi_{\alpha, FM^i}^i, \rho_i^{-1}(\pi_i^{-1}(U_\alpha^i)))\}_{\alpha \in I, i \in \mathbb{N}}$ and $\{(\Psi_\alpha^i, \pi_i^{-1}(U_\alpha^i))\}_{\alpha \in I, i \in \mathbb{N}}$ for FM and TM respectively. It is easy to verify that for $k \leq i \leq j$, $\mathcal{F}^{ik} \circ \mathcal{F}^{ji} = \mathcal{F}^{jk}$ and $\mathcal{H}^{ik} \circ \mathcal{H}^{ji} = \mathcal{H}^{jk}$. Moreover for $j \geq i$, $(\lambda^{ji} \times \lambda^{ji} \times \mathcal{H}^{ji}) \circ \Phi_{\alpha, FM^j}^j = \Phi_{\alpha, FM^i}^i \circ \mathcal{F}^{ji}$ since

$$\begin{aligned} &(\lambda^{ji} \times \lambda^{ji} \times \mathcal{H}^{ji}) \circ \Phi_{\alpha, FM^j}^j([f^j, x^j]^j, (h^1, \dots, h^j)) \\ &= \left(\lambda^{ji} \circ \psi_\alpha^j(x^j), \lambda^{ji} \circ (\psi_\alpha^j \circ f^j)'(0), \mathcal{H}^{ji} \circ (\Psi_{\alpha, x^1}^1 \circ h^1, \dots, \Psi_{\alpha, x^j}^j \circ h^j) \right) \\ &\stackrel{*}{=} \left((\psi_\alpha^i \circ \lambda^{ji})(x^j), (\psi_\alpha^i \circ f^j)'(0); (\Psi_{\alpha, x^1}^1 \circ h^1, \dots, \Psi_{\alpha, x^i}^i \circ h^i) \right) \\ &= \Phi_{\alpha, FM^i}^i \circ \mathcal{F}^{ji}([f^j, x^j]^j, (h^1, \dots, h^j)). \end{aligned}$$

(For * we used the linearity of λ^{ji} and the fact that $\{\psi_\alpha^i\}$ is a projective system of maps.)

3,4. Part 4 is proved and 3 can be proved similarly. The projections $\{\rho_i\}_{i \in \mathbb{N}}$ also form a projective system since for $j \geq i$, $\mathcal{T}^{ji} \circ \rho_j([f^j, x^j]^j; (h^1, \dots, h^j)) = [f^i, x^i]^i$ and $\rho_i \circ \mathcal{F}^{ji}([f^j, x^j]^j; (h^1, \dots, h^j)) = [f^i, x^i]^i$. Consequently

$$\rho = \varprojlim \rho^i; ([f, x], (h^i)_{i \in \mathbb{N}}) \longmapsto [f, x]$$

can be defined and for every $\alpha \in I$, $\Phi_{\alpha, FM} = \varprojlim \Phi_{\alpha, FM^i}$ is a local trivialization for $\rho : FM = \varprojlim FM^i \rightarrow TM = \varprojlim TM^i$. It is easy to check that for $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$,

$$\begin{aligned} \Phi_{\alpha, FM} \circ \Phi_{\beta, FM}^{-1} : U_{\alpha\beta} \times \mathbb{E} \times H_0(\mathbb{F}) &\longrightarrow U_{\alpha\beta} \times \mathbb{E} \times H_0(\mathbb{F}) \\ (\Psi_\beta([f, x]), (h_i)_{i \in \mathbb{N}}) &\longmapsto (\Psi_\alpha([f, x]), (\Psi_{\alpha\beta}^i(x^i) \circ h^i)_{i \in \mathbb{N}}) \end{aligned}$$

i.e. for $x = (x_i)_{i \in \mathbb{N}} \in M$, $\Phi_{\alpha\beta, FM}(x) := \Phi_{\alpha, FM} \circ \Phi_{\beta, FM}^{-1}(x) = \varprojlim \Phi_{\alpha, FM^i}^i \circ \Phi_{\beta, FM^i}^{-1}(x_i)$ is a smooth diffeomorphism. \square

Definition 4.1. $LM := \varprojlim LM^i$ and $FM := \varprojlim FM^i$ are called the generalized frame bundle and spray bundle of $M = \varprojlim M^i$ respectively.

Remarks 2. LM and FM can also be considered as follows

$$LM := \bigcup_{x \in M} \{(h^i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} GL(\mathbb{E}^i, TM_{\mathcal{M}^i(x)} M^i) \text{ s.t. } \varprojlim h^i \text{ exists}\}$$

and

$$FM := \bigcup_{[f, x] \in TM} \{([f, x], (h^i)_{i \in \mathbb{N}}) \in TM \times_M LM\}.$$

For LM and FM consider the following natural actions

$$\begin{aligned} \mathcal{R} : (LM \times \mathbb{F}) \times \mathcal{H}_0(\mathbb{F}) &\longrightarrow LM \times \mathbb{F} \\ \left(((h^i), (v^i)); (g^i) \right)_{i \in \mathbb{N}} &\longmapsto \left((h^i \circ g^i), (g^{i-1} v^i) \right)_{i \in \mathbb{N}} \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}_{FM} : (FM \times \mathbb{F}) \times \mathcal{H}_0(\mathbb{F}) &\longrightarrow FM \times \mathbb{F} \\ \left(\{[f^i, x^i]^i, (h^i)\}, (v^i); (g^i) \right)_{i \in \mathbb{N}} &\longmapsto \left([f^i, x^i]^i, (h^i \circ g^i), (g^{i-1} v^i) \right)_{i \in \mathbb{N}} \end{aligned}$$

then we can prove this generalized theorem.

Proposition 4.2. 1. $\overline{LM} := LM \times \mathbb{F} / \mathcal{H}_0(\mathbb{F})$ is a vector bundle isomorphic to TM .

2. $\overline{FM} := FM \times \mathbb{F} / \mathcal{H}_0(\mathbb{F})$ is a vector bundle isomorphic to $\pi^* TM = \varprojlim \pi_i^* TM^i$

Proof. We prove 2. and 1 can be proved analogously. Define the map

$$\tilde{\rho} : (FM \times \mathbb{F}) / \mathcal{H}_0(\mathbb{F}) \longrightarrow TM; [(\{[f^i, x^i]^i, (h^i)\}, (v^i))_{i \in \mathbb{N}}] \longmapsto ([f^i, x^i]^i)_{i \in \mathbb{N}}$$

Clearly $\tilde{\rho}$ is well defined. Now we prove that $\tilde{\rho}$ is a vector bundle. For an open cover $\{(U_\alpha, \psi_\alpha)\}_{i \in \mathbb{N}, \alpha \in I}$ of M we construct a family of trivializations (charts) for $\tilde{\rho}$ as follows. For $\alpha \in I$ define

$$\begin{aligned} \tilde{\Phi}_\alpha : \tilde{\rho}^{-1}(\pi^{-1}(U_\alpha)) &\longrightarrow (\psi_\alpha(U_\alpha) \times \mathbb{F}) \times \mathbb{F} \\ [(\{[f^i, x^i]^i, (h^i)\}, (v^i))_{i \in \mathbb{N}}] &\longmapsto (\Psi_\alpha^i([f^i, x^i]^i), \Psi_\alpha^i(x^i) \circ h^i(v^i))_{i \in \mathbb{N}}. \end{aligned}$$

$\tilde{\Phi}_\alpha$ is well defined since for any $g = (g^i)_{i \in \mathbb{N}} \in H_0(\mathbb{F})$, and any class $[(\{[f^i, x^i]^i, (h^i)\}, (v^i))_{i \in \mathbb{N}}] \in \overline{FM}$,

$$\begin{aligned} \tilde{\Phi}_\alpha \left([(\{[f^i, x^i]^i, (h^i \circ g^i)\}, (g^{i-1} v^i))_{i \in \mathbb{N}}] \right) &= (\Psi_\alpha^i([f^i, x^i]^i), \Psi_\alpha^i(x^i) \circ h^i(v^i))_{i \in \mathbb{N}} \\ &= \tilde{\Phi}_\alpha \left([(\{[f^i, x^i]^i, (h^i)\}, (v^i))_{i \in \mathbb{N}}] \right). \end{aligned}$$

Let $\tilde{\Phi}_\alpha \left([(\{[f^i, x^i]^i, (h^i)\}, (v^i))_{i \in \mathbb{N}}] \right) = \tilde{\Phi}_\alpha \left([(\{[\bar{f}^i, \bar{x}^i]^i, (\bar{h}^i)\}, (\bar{v}^i))_{i \in \mathbb{N}}] \right)$, then for any $i \in \mathbb{N}$, $[f^i, x^i]^i = [\bar{f}^i, \bar{x}^i]^i$ which means that $[f, x] = \{[f^i, x^i]^i\}_{i \in \mathbb{N}} = \{[\bar{f}^i, \bar{x}^i]^i\}_{i \in \mathbb{N}} = [\bar{f}, \bar{x}]$. Hence h and \bar{h} are generalized linear frames at x . The generalized toplinear (linear isomorphism) map $g := \varprojlim h^{i-1} \circ \bar{h}^i$ exists and $[(\{[f^i, x^i]^i, (h^i)\}, (v^i))_{i \in \mathbb{N}}] = [(\{[f^i, x^i]^i, (\bar{h}^i)\}, (\bar{v}^i))_{i \in \mathbb{N}}]$ i.e. $\tilde{\Phi}_\alpha$ is injective.

$\tilde{\Phi}_\alpha$ is also surjective since for $(\Psi_\alpha([f, x]), v) \in (\psi_\alpha(U_\alpha) \times \mathbb{F}) \times \mathbb{F}$,

$$\tilde{\Phi}_\alpha \left([(\{[f^i, x^i]^i, (\Psi_\alpha^i(x^i)^{-1})\}, v^i)_{i \in \mathbb{N}}] \right) = (\Psi_\alpha([f, x]), v)$$

Let $\alpha, \beta \in I$ s.t. $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ then

$$\begin{aligned} \tilde{\Phi}_\alpha \circ \tilde{\Phi}_\beta^{-1} : (\psi_\beta(U_{\alpha\beta}) \times \mathbb{F}) \times \mathbb{F} &\longrightarrow (\psi_\alpha(U_{\alpha\beta}) \times \mathbb{F}) \times \mathbb{F} \\ (\Psi_\beta([f, x]), v) &\longmapsto (\Psi_\alpha([f, x]), \Psi_{\alpha, x} \circ \Psi_{\beta, x}^{-1}(v)) \end{aligned}$$

Hence $\tilde{\Phi}_{\alpha\beta}([f, x]) := \tilde{\Phi}_\alpha \circ \tilde{\Phi}_\beta^{-1}([f, x]) = \varprojlim \tilde{\Phi}_{\alpha\beta}^i([f^i, x^i]) = \Psi_{\alpha\beta}(x)$ which proves that $\tilde{\rho}$ is a generalized vector bundle with fibres of type \mathbb{F} and the structure group $\mathcal{H}_0(\mathbb{F})$. (see also [2], [11]).

Define

$$\begin{aligned} \Xi_{FM} : FM \times \mathbb{F}/\mathcal{H}_0(\mathbb{F}^i) &\longrightarrow \pi^*TM \\ \left(\left([f^i, x^i]^i, h^i, v^i \right)_{i \in \mathbb{N}} \right) &\longmapsto \left([f^i, x^i]^i, [\bar{f}^i, x^i]^i \right)_{i \in \mathbb{N}} \end{aligned}$$

where $\bar{f}^i(t) := \psi_\alpha^{i-1}(x^i + th^i(v^i))$. Note that $\{\bar{f}^i\}_{i \in \mathbb{N}}$ forms a projective system of curves in M . In fact $\bar{f} = \varprojlim \bar{f}^i$ represents the tangent vector $h(v)$ at the base point x .

Ξ_{FM} is well defined since for any $g = (g^i) \in \mathcal{H}_0(\mathbb{F})$,

$$\Xi_{FM} \left(\left([f^i, x^i]^i, h^i \circ g^i, g^{i-1} v^i \right)_{i \in \mathbb{N}} \right)$$

coincides with $\Xi_{FM} \left(\left([f^i, x^i]^i, h^i, v^i \right)_{i \in \mathbb{N}} \right) = ([f^i, x^i]^i, [\bar{f}^i, x^i]^i)_{i \in \mathbb{N}}$. To see that Ξ_{FM} is injective, suppose that

$$\Xi_{FM} \left(\left([f^i, x^i]^i, h^i, v^i \right)_{i \in \mathbb{N}} \right) = \Xi_{FM} \left(\left([\bar{f}^i, x^i]^i, \bar{h}^i, \bar{v}^i \right)_{i \in \mathbb{N}} \right).$$

Since $[f, x] = [\bar{f}, \bar{x}]$ we conclude that h and \bar{h} are generalized frame at $x = \bar{x}$. Furthermore $\left(\left([f^i, x^i]^i, h^i, v^i \right)_{i \in \mathbb{N}} \right) = \left(\left([\bar{f}^i, \bar{x}^i]^i, \bar{h}^i, \bar{v}^i \right)_{i \in \mathbb{N}} \right)$ are equivalent, w.r.t. the action \mathcal{R}_{FM} , via the generalized toplinear map $g := h^{-1} \circ \bar{h} = \varprojlim h^{i-1} \circ \bar{h}^i$.

Suppose that $([f, x], [\bar{f}, \bar{x}]) \in \pi^*TM$ be arbitrary with $x = (x^i)_{i \in \mathbb{N}} \in U_\alpha$ then

$$\Xi_{FM} \left(\left([f^i, x^i]^i, \Psi_{\alpha, x^i}^{i-1}; pr_2 \circ \Psi_\alpha^i [f^i, x^i]^i \right)_{i \in \mathbb{N}} \right) = ([f^i, x^i]^i, [\bar{f}^i, x^i]^i)_{i \in \mathbb{N}}.$$

Since $\{M^i\}_{i \in \mathbb{N}}$ and consequently $\{TM^i\}_{i \in \mathbb{N}}$ are projective systems of manifolds, then for every $x = (x_i)_{i \in \mathbb{N}}$, $\{\Psi_\alpha^i(x^i)\}_{i \in \mathbb{N}}$ forms a projective system of maps and $\{\pi^{i*}TM^i\}_{i \in \mathbb{N}}$ is also a projective system of vector bundles over $\{TM^i\}_{i \in \mathbb{N}}$ [2].

Finally we show that Ξ_{FM} maps trivializations of $\tilde{\rho}$ to trivializations of π^* which makes Ξ_{FM} a vector bundle isomorphism. For a limit chart (U_α, ψ_α) of M consider the corresponding limit charts $(\pi^{-1}(U_\alpha), \Psi_\alpha)$ and $(\rho^{-1}(\pi^{-1}(U_\alpha)), \tilde{\Phi}_\alpha)$ for $TM = \varprojlim TM^i$ and $FM = \varprojlim FM^i$ respectively. As a result, we see that for any $([f, x], h, v) \in \overline{FM}$;

$$\begin{aligned} &\Psi_\alpha \times_M \Psi_\alpha \circ \Xi_{FM} \left(([f, x], h, v) \right) \\ &= \{ \Psi_\alpha^i \times_{M^i} \Psi_\alpha^i \circ \Xi_{FM^i} \left(([f^i, x^i]^i, h^i, v^i) \right) \}_{i \in \mathbb{N}} \\ &= \{ \Psi_\alpha^i \times_{M^i} \Psi_\alpha^i \left(([f^i, x^i]^i, [\bar{f}^i, x^i]^i) \right) \}_{i \in \mathbb{N}} \\ &= \{ \tilde{\Phi}_\alpha^i \left(([f^i, x^i]^i, h^i, v^i) \right) \}_{i \in \mathbb{N}} \\ &= \tilde{\Phi}_\alpha \left(([f, x], h, v) \right) \end{aligned}$$

which completes the proof. □

4.1. Pre-Finsler Parallelism and ordinary differential equations on Fréchet manifolds

In spite of the lack of a general solvability-uniqueness theorem for ordinary differential equations on non-Banach manifolds, and even spaces, we state the following existence and uniqueness theorem for ordinary differential equations on projective limit manifolds. Let $\{M^i\}_{i \in \mathbb{N}}$ be a projective system of Banach manifolds with the limit $M = \varprojlim M^i$ and $c = \varprojlim c^i : (-\epsilon, \epsilon) \rightarrow M$ be a smooth curve in M obtained as the limit of curves $\{c^i\}_{i \in \mathbb{N}}$.

Theorem 4.1. *If (Γ, D) is a projective limit pre-Finsler connection on M then, for every $\xi_0 = (x_0, y_0, z_0) \in \psi_\alpha(U_\alpha) \times \mathbb{F} \times \mathbb{F}$ there exists a unique (Γ, D) -horizontal lift for c say \bar{C} with $\bar{C}(0) = \xi_0$.*

Proof. Suppose that $(U_\alpha^i, \psi_\alpha^i)$ is a projective limit chart around $x_0 = (x_0^i)_{i \in \mathbb{N}}$. For any $i \in \mathbb{N}$, c^i is a curve in M^i , D^i is a connection on M^i and $(x_0^i, y_0^i) \in \psi_\alpha^i(U_\alpha^i) \times \mathbb{E}^i$. Then according to the theorem 3.1 locally there exists a unique curve $C^i(t) = (x^i(t), y^i(t)) : (-\epsilon, \epsilon) \rightarrow M^i$ which is D^i -horizontal lift of c^i and satisfies the equation;

$$y^{i'}(t) + \omega_\alpha^i(x^i(t), y^i(t))x^{i'}(t) = 0 \quad (4.1)$$

such that $C^i(0) = (x_0^i, y_0^i)$. The idea of the proof is to show that $\{C^i\}_{i \in \mathbb{N}}$ form a projective system of curves and $C = \varprojlim C^i$ is the desired, D -horizontal lift, curve. Note that here the domain of C^i may depends on i i.e. $C^i : (-\epsilon^i, \epsilon^i) \rightarrow M^i$ and $\lim_{i \rightarrow \infty} \epsilon^i = 0$. In this case the solution is trivial. To avoid this we need some Lipschitz condition on local components as it is stated in appendix of [1].

For $j \geq i$, we claim that $\mathcal{T}^{j_i} \circ C^j = C^i$, or locally $\lambda^{j_i} \times \lambda^{j_i} \circ (x^j(t), y^j(t)) = (x^i(t), y^i(t))$, and consequently $C = \varprojlim C^i$ exists as a curve in E .

$$\begin{aligned} & (\lambda^{j_i} \circ y^j)'(t) + \omega_\alpha^i((\lambda^{j_i} \circ x^j)(t), (\lambda^{j_i} \circ x^j)'(t))[(\lambda^{j_i} \circ y^j)(t)] \\ &= \lambda^{j_i} \circ y^{j'}(t) + \omega_\alpha^i((\lambda^{j_i} \circ x^j)(t), \lambda^{j_i} \circ x^{j'}(t))[(\lambda^{j_i} \circ y^j)(t)] \\ &= \lambda^{j_i} \circ (y^{i'}(t) + \omega_\alpha^i(x^i(t), y^i(t))x^{i'}(t)) = 0 \end{aligned}$$

Furthermore for $j \geq i$, $\lambda^{j_i} \times \lambda^{j_i} \circ (x^j(0), y^j(0)) = \lambda^{j_i} \times \lambda^{j_i}(x_0^j, y_0^j) = (x_0^i, y_0^i)$. Using uniqueness theorem for ordinary differentia equations on Banach manifolds we conclude that $\lambda^{j_i} \times \lambda^{j_i} \circ (x^j(t), y^j(t)) = (x^i(t), y^i(t))$ or locally near (x_0^i, y_0^i) , $\mathcal{T}^{j_i} \circ C^j = C^i$. Hence $\{C^i\}_{i \in \mathbb{N}}$ form a projective system of curves and $C = \varprojlim C^i$ is a curve in $TM = \varprojlim TM^i$. Moreover

$$y'(t) + \omega_\alpha(x(t), y(t))x'(t) = (y^{i'}(t) + \omega_\alpha^i(x^i(t), y^i(t))x^{i'}(t))_{i \in \mathbb{N}} = 0$$

and clearly $C(0) = (x(0), y(0)) = ((x^i(0), y^i(0)))_{i \in \mathbb{N}} = \xi_0$.

On the other hand $p \circ C = (p^i \circ C^i)_{i \in \mathbb{N}} = (c^i)_{i \in \mathbb{N}} = c$ i.e. C is D -horizontal lift of c . Let C_1 be another curve above c with $D \circ C_1'(t) = 0$ i.e. is a D -horizontal lift of c . Then for every $i \in \mathbb{N}$ using the canonical projection $\mathcal{T}^i : TM \rightarrow TM^i$ we obtain the family of curves $\{C_1^i\}$. However for any $i \in \mathbb{N}$, C_1^i is another solution for 4.1 which satisfy the same boundary conditions. Hence For every integer i , $C_1^i = C^i$ and consequently $C_1 = \varprojlim C_1^i = \varprojlim C^i = C$. \square

Hence, we proved that for the curve c in M there exists a unique D -horizontal lift C in TM . In a same way one can prove that for C there exists a unique lift \bar{C} in FM which is Γ horizontal. As a result \bar{C} is a (Γ, D) horizontal lift of c with the given boundary conditions.

5. Applications and examples

In this section we state two examples to support our theory.

5.1. Infinite jets

Let (F, π, M) be a finite dimensional fibre bundle with $\dim M = m$ and $\dim F = m + n$. Consider an atlas of adopted coordinates for F i.e. if (U, Φ) is a local chart around $a \in F$ then $pr_1 \circ \Phi = x \circ \pi$ where x is a chart coordinate around $\pi(a) \in M$ and $pr_1 : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m$ is projection to the first m factors.. We denote the space of the local sections of π with $\Gamma_p(\pi)$ for $p \in M$.

For two local sections ξ and η in $\Gamma_p(\pi)$ we say that they are 1-equivalent if $\xi(p) = \eta(p)$ and for some (and consequently all) adopted coordinate $\Phi = (x^i, \phi^\alpha)$ around $\xi(p)$

$$\frac{\partial \xi^\alpha}{\partial x^i} = \frac{\partial \eta^\alpha}{\partial x^i}$$

for $i = 1, \dots, m$ and $\alpha = 1, \dots, n$. (Here $\xi^\alpha = \phi^\alpha \circ \xi$ and (x^i) is a local chart for M around p). The equivalence class containing ξ is called the first order jet ξ at p and is denoted by $j_p^1 \xi$. The first order jet manifold of the fibre bundle π is $J^1 \pi := \{j_p^1 \xi; p \in M \text{ and } \xi \in \Gamma_p(\pi)\}$ is an $m + n + mn$ dimensional manifold. The canonical local chart for $j^1 \pi$ with respect to the adopted coordinate (U, Φ) is (U^1, Φ^1) where $U^1 = \{j_p^1 \xi; \xi(p) \in U\}$ and $\Phi^1 = (x^i, \phi^\alpha, \phi_i^\alpha)$ with

$$x^i(j_p^1 \xi) = x^i(p), \phi^\alpha(j_p^1 \xi) = \phi^\alpha(\xi(p)), \phi_i^\alpha(j_p^1 \xi) = \frac{\partial \xi^\alpha}{\partial x^i} \Big|_p.$$

In the same way we can define $J^2\pi := \{j_p^2\xi; p \in M \text{ and } \xi \in \Gamma_p(\pi)\}$ using the following equivalence relation on the local sections

$$j_p^2\xi = j_p^2\eta \iff \xi(p) = \eta(p), \frac{\partial \xi^\alpha}{\partial x^i} = \frac{\partial \eta^\alpha}{\partial x^i}, \frac{\partial^2 \xi^\alpha}{\partial x^i \partial x^j} = \frac{\partial^2 \eta^\alpha}{\partial x^i \partial x^j}.$$

In this way $J^2\pi$ becomes a smooth manifold with canonical atlas of charts (U^2, Φ^2) where $U^2 = \{j_p^2\xi; \xi(p) \in U\}$, $\Phi^2 = (x^i, \phi^\alpha, \phi_i^\alpha, \phi_{ij}^\alpha)$ and

$$\phi_{ij}^\alpha : j_p^2\xi \mapsto \frac{\partial^2 \xi^\alpha}{\partial x^i \partial x^j}.$$

Using multi index notation ([21] chapter 8) for any natural number $k \geq 2$ we define the k -th order jet manifold, $J^k\pi$ with the equivalence relation

$$j_p^k\xi = j_p^k\eta \iff \xi(p) = \eta(p), \frac{\partial^{|I|} \xi^\alpha}{\partial x^{|I|}} = \frac{\partial^{|I|} \eta^\alpha}{\partial x^{|I|}}$$

for every multi-index with $1 \leq |I| \leq k$. $J^k\pi$ is also a smooth finite dimensional manifold with local charts $U^k = \{j_p^k\xi; \xi(p) \in U\}$, $\Phi^k = (x^i, \phi^\alpha, \phi_{|I|}^\alpha)$. (For a detailed study about the jet bundles see [21].)

For any $k \in \mathbb{N}$, the map $\pi_{k+1,k} : J^{k+1}\pi \rightarrow J^k\pi$; $j_p^{k+1}\xi \mapsto j_p^k\xi$ is surjective submersion. For any $k \in \mathbb{N}$ let $\dim J^k\pi = N(k)$ and $\lambda_{k+1,k} : \mathbb{R}^{N(k+1)} \rightarrow \mathbb{R}^{N(k)}$ be the natural projection to the first $N(k)$ -components. With these notations the family $\{J^k\pi, \pi_{k+1,k}\}_{k \in \mathbb{N}}$ forms a projective system (inverse system) of finite dimensional manifolds modeled on $\{\mathbb{R}^{N(k)}, \lambda_{k+1,k}\}_{k \in \mathbb{N}}$. Define $J^\infty\pi := \varprojlim J^k\pi$. Then $J^\infty\pi$ is a Fréchet manifold modeled on the Fréchet space $\mathbb{R}^\infty = \varprojlim \mathbb{R}^k$. More precisely $J^\infty\pi$ is a subset of $\prod_{k \in \mathbb{N}} J^k\pi$ consisting strings of the form $(j_p^k\xi)_{k \in \mathbb{N}}$ for $p \in M$ and $\xi \in \Gamma_p(\pi)$ and $(U^k, \Phi^k)_{k \in \mathbb{N}}$ is a projective system of charts with the limit $(U^\infty = \varprojlim U^k, \Phi^\infty = \varprojlim \Phi^k)$ as a projective limit chart for $J^\infty\pi$. Clearly the collection of these charts forms an atlas modelling $J^\infty\pi$ on \mathbb{R}^∞ .

According to our approach, the following statements hold true for the infinite jet manifold $J^\infty\pi$.

1. Lemma 4.1 states that for any $k \in \mathbb{N}$;

$$LJ^k\pi := \bigcup_{j_p^k\xi \in J^k\pi} \{(h^i)_{1 \leq i \leq k} \in \prod_{i=1}^k Lis(\mathbb{R}^{N(i)}, T_{j_p^i\xi} J^i\pi) \text{ s.t.} \\ \mathcal{T}^{j^i} \circ h^j = h^i \circ \lambda^{j^i} \text{ for } 1 \leq i \leq j \leq k\}$$

is a principal fibre bundle on $J^k\pi$ with the structure group

$$\mathcal{H}_0^k(\mathbb{R}^{N(k)}) := \{(g^i)_{1 \leq i \leq k} \in \prod_{i=1}^k GL(\mathbb{R}^{N(i)}) : \lambda^{j^i} \circ g^j = g^i \circ \lambda^{j^i} \text{ for all } 1 \leq i \leq j \leq k\}.$$

2. For any $k \in \mathbb{N}$;

$$FJ^k\pi := \bigcup_{j_p^k\xi \in J^k\pi} \{([f^k, j_p^k\xi]^k; (h^1, \dots, h^k)) \text{ s.t. } [f^k, j_p^k\xi]^k \in T_{j_p^k\xi} J^k\pi, \\ h^i \in Lis(\mathbb{R}^{N(i)}, T_{j_p^i\xi} J^i\pi) \text{ and } \mathcal{T}^{j^i} \circ h^j = h^i \circ \lambda^{j^i} \text{ for } 1 \leq i \leq j \leq k\}$$

is a principal fibre bundle on $TJ^k\pi$ with the structure group $\mathcal{H}_0^k(\mathbb{R}^{N(k)})$.

3. $\{LJ^k\pi, \mathcal{H}_0^k(\mathbb{R}^{N(k)}), J^k\pi, p^k\}_{k \in \mathbb{N}}$ and $\{FJ^k\pi, \mathcal{H}_0^k(\mathbb{R}^{N(k)}), TJ^k\pi, \rho^k\}_{k \in \mathbb{N}}$ are projective system of principal fibre bundles and

$$LJ^\infty\pi = \varprojlim LJ^k\pi \text{ and } FJ^\infty\pi = \varprojlim FJ^k\pi$$

are generalized principal bundles on $J^\infty\pi$ and $TJ^\infty\pi$ respectively.

4. Suppose that we have a projective system of Riemannian metrics $\{g^k\}_{k \in \mathbb{N}}$ on $\{J^k\pi\}_{k \in \mathbb{N}}$ then one can construct a projective system of connections $\{\nabla^k\}_{k \in \mathbb{N}}$ with the limit $\nabla^\infty = \varprojlim \nabla^k$ as a generalized connection on $J^\infty\pi$. The existence and uniqueness of ∇^∞ -horizontal lift is a direct consequence of theorem 4.1. Moreover one can directly consider a family of (possibly nonlinear) connection which are not necessarily Levi-Civita connections and deduce the same results. (For an example of such a connection on $J^\infty\pi$ see [15])

5.2. Manifold of mappings

Let N be a compact finite dimensional smooth manifold and M a Banach manifold without boundary admitting a connection ∇ . Suppose that $C^k(N, M)$, $k \geq 1$, denote the space of C^k functions between manifolds N and M . For a C^k map $h : N \rightarrow M$ we use the exponential map of ∇ to model $C^k(N, M)$ on the Banach (or Sobolev) space of C^k sections $C^k(h^*TM)$ (Here h^*TM is the pullback bundle of TM via h).

More precisely if $exp : \mathcal{O} \rightarrow M$, $\mathcal{O} \subseteq TM$, is the exponential map corresponding to ∇ on M and $\mathcal{D} \subseteq \mathcal{O}$ an open neighborhood of the set of zero vectors in TM such that $(\pi_M, exp)|_{\mathcal{D}}$ is a diffeomorphism then

$$C^k(exp) : C^k(h^*\mathcal{D}) \rightarrow C^k(N, M); \xi \mapsto exp \circ \xi$$

is a chart for $C^k(N, M)$ called the natural chart centered at h [9]. A collection of these charts form an atlas for $C^k(N, M)$ say $\mathcal{C}^k := \{(U_h^k, \phi_h^k)\}$.

The family $\{C^k(N, M)\}_{k \in \mathbb{N}}$ forms a projective system of manifolds with the connecting morphisms, for $k \geq j$, $\mathcal{K}^{kj} : C^k(N, M) \rightarrow C^j(N, M)$ inclusions as well as the connecting morphism for the model spaces.

Eliasson in [9] theorem 5.4 proved that every connection ∇ on M induces a natural connection $C^K(\nabla)$ on $C^k(N, M)$.

By these means on the projective system of Banach manifolds $\{C^k(N, M)\}_{k \in \mathbb{N}}$ we have a projective system of connections $\{C^k(\nabla)\}_{k \in \mathbb{N}}$ with the limit $C^\infty(\nabla) = \varprojlim C^k(\nabla)$ which is a limit connection on the Fréchet manifold $C^\infty(N, M)$.

Note that one can use the benefits of our method in the Eliasson's framework to study partial differential equations. Moreover one can apply our technique to study ordinary differential equations the group of diffeomorphisms [20].

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