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A Curvature Energy Problem on a Timelike Surface

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ABSTRACT

We present a variational study of finding null relaxed elastic lines which are extremals of a geometric energy functional, subject to suitable constraints and boundary conditions on a timelike surface in Minkowski 3-space. We derive an Euler-Lagrange equation for a null relaxed elastic line with regard to geodesic curvature, geodesic torsion and normal curvature of the curve on the timelike surface. Finally, we give some examples for null relaxed elastic lines on the pseudo-sphere and pseudo-cylinder.

Keywords: *Null relaxed elastic line; Euler-Lagrange equation; variational calculus.*

AMS Subject Classification (2010): *Primary: 53A35 ; Secondary: 53B30; 35A15.*

1. Introduction

A relaxed elastic line introduced by Manning in [5], minimizes the total squared curvature functional $\int_0^\ell \kappa^2(s)ds$ among curves of length ℓ with fixed initial point and direction. Nickerson and Manning in [6] derive the intrinsic equations for a relaxed elastic line on an oriented surface in Euclidean 3-space. They characterize the relaxed elastic line with the following differential equation

$$2\kappa_g'' - 4\tau_g\kappa_n' - 2\kappa_n\tau_g' + \kappa_g\left(\kappa_g^2 + \kappa_n^2 - 2\tau_g^2 + \kappa_n^2(\ell)\right) = 0$$

with two boundary conditions

$$\begin{aligned} &\kappa_g(\ell) = 0 \\ &\kappa_g'(\ell) = 2\kappa_n(\ell)\,\tau_g(\ell)\,. \end{aligned}$$

They give necessary and sufficent condition that an arc of a geodesic is a relaxed elastic line on a general surface. They also show that the geodesics of some surfaces in Euclidean 3-space are relaxed elastic lines. Yücesan etc. derive the Euler-Lagrange equation for a non-null relaxed elastic line on pseudo-hypersurfaces in pseudo-Euclidean spaces and solve this equations for some pseudo-hypersurfaces as pseudo-hyperplane, pseudo-hypersphere, pseudo-hyperbolic space and pseudo-hypercylinder [9].

Although non-null relaxed elastic lines are studied on some surfaces in Minkowski $3-$space, null ones are not. So we focus the problem of finding null relaxed elastic line on a timelike surface in Minkowski $3-$space whose unit normal vector field is everywhere spacelike. Thus we construct the problem of finding null relaxed elastic line and give a differential condition which is derived together with boundary conditions that must be satisfied for any null relaxed elastic line in a timelike surface. These conditions are examined in the special cases of pseudo-sphere and pseudo-cylinder which are timelike surfaces.

2. Geometrical Set Up

Recall that the Minkowski $3-$space $\mathbb{E}_1^3$ is a three-dimensional real vector space equipped with the metric

$$< x, y >= -x_1y_1+x_2y_2+x_3y_3, \quad x = (x_1, x_2, x_3)\,, y = (y_1, y_2, y_3) \in \mathbb{E}_1^3$$

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which is a non-degenerate, symmetric and bilinear form. A smooth curve in $\mathbb{E}_1^3$ is spacelike (resp., timelike and null), if its velocity vector is spacelike (resp., timelike and null). If a regular curve in $\mathbb{E}_1^3$ is spacelike or timelike, we can reparametrize the curve by an arc-length. For null (or lightlike) curves, there would be not sense reparametrize by the arc-length. However, they have pseudo arc-length parametrized. A surface in $\mathbb{E}_1^3$ is a non-degenerate (or a degenerate) if induced metric on its tangent plane is non-degenerate (or degenerate). A non-degenerate surface is named in terms of the induced metric. If the induced metric is positive definite, a non-degenerate surface is called spacelike, if the induced metric is indefinite, a non-degenerate surface is called timelike (see [4] and [7]).
Now, we consider an oriented timelike surface S in Minkowski $3-$space $\mathbb{E}_1^3$ and let γ be a null curve on S. At a point $\gamma(s)$ of γ, let $T(s) = \gamma^{'}(s)$ denote the tangent vector to γ, let $n(s)$ denote the spacelike unit normal vector to S and let $Q(s)$ denote the unique vector obtained by

$$Q = \frac{1}{< V, T >}\{V - \frac{< V, V >}{2 < V, T >}T\}, V \in T_{\gamma(s)}S, \ \ < V, T >\neq 0$$

which appears in [2] such that

$$< T, T >=< Q, Q >=< T, n >=< Q, n >= 0,$$
$$< T, Q >=< n, n >= 1.$$

For each s, $\{T, Q, n\}$ is called the Darboux frame along γ. The derivative equations of Darboux frame are given by

$$\begin{pmatrix} T' \\ Q' \\ n' \end{pmatrix} = \begin{pmatrix} \kappa_g & 0 & \kappa_n \\ 0 & -\kappa_g & \tau_g \\ -\tau_g & -\kappa_n & 0 \end{pmatrix} \begin{pmatrix} T \\ Q \\ n \end{pmatrix}, \quad (2.1)$$

where κ_g, $\kappa_n =< A_nT, T >$ and $\tau_g =< A_nT, Q >$ are the geodesic curvature, the normal curvature and the geodesic torsion of γ, respectively, and A_n is the shape operator of S [1]. On the other hand, we have the Cartan frame $\{T, N, B\}$ for γ a null curve parametrized by the pseudo-arc parameter in Minkowski $3-$space $\mathbb{E}_1^3$, where $T(s) = \gamma'(s)$, $N(s) = \gamma''(s)$ is a unit spacelike vector field and $B(s) = -\gamma''' - \frac{1}{2} < \gamma''', \gamma''' > \gamma'$ such that satisfies

$$< T, T >=< B, B >=< T, N >=< N, B >= 0,$$
$$< N, N >=< T, B >= 1$$

[3]. The derivative equations of Cartan frame are given by

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\kappa & 0 & -1 \\ 0 & \kappa & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \quad (2.2)$$

where κ is the lightlike curvature of γ with the equality

$$\kappa(s) = \frac{1}{2} < \gamma''', \gamma''' >$$

[3]. We have the following equations from the comparison of moving frames (2.1) and (2.2)

$$< N, n >= \kappa_n = \pm 1$$

(see [1]).

3. Minimizing the Functional

Any null relaxed elastic line on a timelike surface S is a critial curve of the functional

$$K = \int_0^\ell \kappa^2 ds = \frac{1}{4}\int_0^\ell < \gamma''', \gamma''' >^2 ds \quad (3.1)$$

among the family of all arc of length ℓ having the same initial point and direction with $\gamma \in S$. Now we suppose that γ lies a coordinate patch

$$X(u,v) = (x(u,v), y(u,v), z(u,v)).$$

So, the tangent vector to

$$\gamma(s) = X(u(s), v(s))$$

is expressed with

$$T(s) = \gamma'(s) = \frac{du}{ds}X_u + \frac{dv}{ds}X_v$$

and for suitable scalar functions $p(s)$ and $q(s)$, we can write the binormal vector field of γ

$$Q(s) = p(s)X_u + q(s)X_v.$$

In order to obtain variational arcs of length ℓ, we extend γ to an arc $\gamma^*(s)$ defined for $0 \leq s \leq \ell^*$, with $\ell^* > \ell$ but sufficiently close to ℓ so that γ^* lies in the coordinate patch. Let $\mu(s)$, $0 \leq s \leq \ell^*$, be a scalar not vanishing function. Then, we denote the variation vector field

$$\mu(s)Q(s) = \eta(s)X_u + \zeta(s)X_v$$

along γ. Assume also that

$$\mu(0) = 0, \qquad \mu'(0) = 0. \tag{3.2}$$

No further restrictions will be placed on μ. A variation of γ is defined by

$$\beta(\sigma;t) = X(u(\sigma), v(\sigma)) + t(\eta(\sigma), \zeta(\sigma)), \tag{3.3}$$

for $0 \leq \sigma \leq \ell^*$. Because of (3.2), the variation (3.3) has

$$\beta(0,t) = \gamma(0), \qquad \left.\frac{\partial\beta(\sigma,t)}{\partial\sigma}\right|_{\sigma=0} = \left.\frac{\partial\gamma(\sigma)}{\partial\sigma}\right|_{\sigma=0} = \gamma'(0).$$

It means that the variational arcs has same initial point and initial direction. We can restrict $\beta(\sigma;t)$, $0 \leq |t| < \delta$, to an arc of length ℓ by restricting the parameter σ to an interval of $0 \leq \sigma \leq \lambda(t) \leq \ell^*$ by requiring

$$\int_0^{\lambda(t)} < \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2} >^{\frac{1}{4}} d\sigma = \ell. \tag{3.4}$$

Note that $\lambda(0) = \ell$. The function $\lambda(t)$ need not be determined explicitly, but we shall need its derivative (given in Lemma 3.1). By using (2.1), some partial derivatives of $\beta(\sigma;t)$ with respect to σ are obtained as

$$\left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0} = T,$$

$$\left.\frac{\partial^2\beta}{\partial\sigma^2}\right|_{t=0} = \kappa_g T + \kappa_n n \tag{3.5}$$

and

$$\left.\frac{\partial^3\beta}{\partial\sigma^3}\right|_{t=0} = \left(\kappa_g' + \kappa_g^2 - \kappa_n\tau_g\right)T + \kappa_g\kappa_n n - Q. \tag{3.6}$$

First derivative of $\beta(\sigma;t)$ with respect to t is calculated as

$$\left.\frac{\partial\beta}{\partial t}\right|_{t=0} = \mu Q.$$

Finally, some mixed derivative of $\beta(\sigma;t)$ are obtained as

$$\left.\frac{\partial^2\beta}{\partial\sigma\partial t}\right|_{t=0} = (\mu' - \kappa_g\mu)Q + \mu\tau_g n,$$

$$\left.\frac{\partial^3\beta}{\partial\sigma^2\partial t}\right|_{t=0} = \mu\tau_g^2 T + ((\mu' - \kappa_g\mu)' - (\mu' - \kappa_g\mu)\,\kappa_g - \mu\tau_g\kappa_n)Q + ((\mu' - \kappa_g\mu)\,\tau_g + (\mu\tau_g)')n \tag{3.7}$$

and

$$\begin{aligned}\left.\frac{\partial^4\beta}{\partial\sigma^3\partial t}\right|_{t=0} &= (-\left(\mu\tau_g^2\right)' - \mu\tau_g^2\kappa_g - (\mu' - \kappa_g\mu)\,\tau_g^2 - (\mu\tau_g)'\,\tau_g)T \\ &+((\mu' - \kappa_g\mu)'' - ((\mu' - \kappa_g\mu)\,\kappa_g)' - (\mu\tau_g\kappa_n)' \\ &-\kappa_g\,(\mu' - \kappa_g\mu)' + \kappa_g^2\,(\mu' - \kappa_g\mu) + \mu\tau_g\kappa_g\kappa_n \\ &-\tau_g\kappa_n\,(\mu' - \kappa_g\mu) - \kappa_n\,(\mu\tau_g)')Q \\ &+(-\mu\tau_g^2\kappa_n + (\mu' - \kappa_g\mu)'\,\tau_g - (\mu' - \kappa_g\mu)\,\kappa_g\tau_g \\ &-\mu\tau_g^2\kappa_n + ((\mu' - \kappa_g\mu)\,\tau_g)' + (\mu\tau_g)'')n.\end{aligned} \tag{3.8}$$

Lemma 3.1. *We obtain from (3.4), (3.5) and (3.7) in the following equation*

$$\left.\frac{d\lambda}{dt}\right|_{t=0} = -\frac{1}{2}\int_0^\ell \kappa_g((\mu' - \kappa_g\mu)' - \kappa_g\,(\mu' - \kappa_g\mu) - \mu\tau_g\kappa_n) + \kappa_n((\mu' - \kappa_g\mu)\,\tau_g + (\mu\tau_g)')ds. \tag{3.9}$$

Let K denote the null relaxed elastic functional of arc $\beta(\sigma;t)$, $0 \leq \sigma \leq \lambda(t)$. A necessary condition that γ be an extremal is that $K'(0) = 0$ for arbitrary μ satisfying (3.2) (see [8]). If we calculate $\frac{dK}{dt}$, we get

$$\frac{dK}{dt} = \frac{1}{4}\left.\frac{d\lambda}{dt}\right|_{t=0}\left\{<\frac{\partial^3\beta}{\partial\sigma^3},\frac{\partial^3\beta}{\partial\sigma^3}>\right\}_{\sigma=\lambda(t)} + \int_0^{\lambda(t)}<\frac{\partial^3\beta}{\partial\sigma^3},\frac{\partial^3\beta}{\partial\sigma^3}><\frac{\partial^4\beta}{\partial\sigma^3\partial t},\frac{\partial^3\beta}{\partial\sigma^3}>d\sigma. \tag{3.10}$$

Substituting (3.9), (3.6) and (3.8) in the equation (3.10), and then if we write zero in place of t in the functional, we obtain;

$$\begin{aligned}\left.\frac{dK}{dt}\right|_{t=0} = &\int_0^\ell \mu''\,(A\kappa_g + BC\,(1 - 3\kappa_g) + 3B\kappa_g\kappa_n\tau_g)\,ds \\ &+\int_0^\ell \mu'(A(-2\kappa_g^2 + 2\kappa_n\tau_g) + 3BC\left(-\kappa_g' + \kappa_g^2 - \kappa_n\tau_g\right) \\ &+B(3\tau_g^2 - 3\kappa_g^2\kappa_n\tau_g + 3\tau_g'\kappa_n\kappa_g)ds \\ &+\int_0^\ell \mu(A(-\kappa_g\kappa_g' + \kappa_g^3 - 2\kappa_g\kappa_n\tau_g + \kappa_n\tau_g') + BC(-\kappa_g'' \\ &+3\kappa_g\kappa_g' - 2\tau_g'\kappa_n - \kappa_g^3 + 2\kappa_g\kappa_n\tau_g) + B(3\tau_g\tau_g' - 2\tau_g^2\kappa_g \\ &-2\kappa_g\kappa_g'\kappa_n\tau_g + \kappa_g^3\kappa_n\tau_g - \kappa_g^2\kappa_n\tau_g' + \kappa_g\kappa_n\tau_g'')ds,\end{aligned}$$

where

$$A = \frac{1}{8}\left(2\kappa_g'(\ell) + \kappa_g^2(\ell) - 2\kappa_n(\ell)\,\tau_g(\ell)\right),$$

$$B = -\left(2\kappa_g' + \kappa_g^2 - 2\kappa_n\tau_g\right)$$

and

$$C = \kappa_g' + \kappa_g^2 - \kappa_n\tau_g.$$

Integrating by parts, we get

$$\begin{aligned}\left.\frac{dK}{dt}\right|_{t=0} = &\int_0^\ell \mu(A(-\kappa_g\kappa_g' + \kappa_g^3 - 2\kappa_g\kappa_n\tau_g + \kappa_n\tau_g') + BC(-\kappa_g'' \\ &+3\kappa_g\kappa_g' - 2\tau_g'\kappa_n - \kappa_g^3 + 2\kappa_g\kappa_n\tau_g) + B(3\tau_g\tau_g' - 2\tau_g^2\kappa_g \\ &-2\kappa_g\kappa_g'\kappa_n\tau_g + \kappa_g^3\kappa_n\tau_g - \kappa_g^2\kappa_n\tau_g' + \kappa_g\kappa_n\tau_g'' - (A(-2\kappa_g^2 \\ &+2\kappa_n\tau_g) + 3BC\left(-\kappa_g' + \kappa_g^2 - \kappa_n\tau_g\right) + B(3\tau_g^2 - 3\kappa_g^2\kappa_n\tau_g \\ &+3\tau_g'\kappa_n\kappa_g))' + (A\kappa_g + BC\,(1 - 3\kappa_g) + 3B\kappa_g\kappa_n\tau_g)'')ds \\ &+u_1(\ell)\,\mu'(\ell) + (u_2(\ell) - u_1'(\ell))\,\mu(\ell)\end{aligned}$$

where

$$\begin{aligned} u_1(\ell) = \; & A(-\kappa_g(\ell)\kappa_g'(\ell) + \kappa_g^3(\ell) - 2\kappa_g(\ell)\kappa_n(\ell)\tau_g(\ell) \\ & + \kappa_n(\ell)\tau_g'(\ell)) + B(\ell)C(\ell)(-\kappa_g''(\ell) + 3\kappa_g(\ell)\kappa_g'(\ell) \\ & - 2\tau_g'(\ell)\kappa_n(\ell) - \kappa_g^3(\ell) + 2\kappa_g(\ell)\kappa_n(\ell)\tau_g(\ell)) \\ & + B(\ell)(3\tau_g(\ell)\tau_g'(\ell) - 2\tau_g^2(\ell)\kappa_g(\ell) \\ & - 2\kappa_g(\ell)\kappa_g'(\ell)\kappa_n(\ell)\tau_g(\ell) + \kappa_g^3(\ell)\kappa_n(\ell)\tau_g(\ell) \\ & - \kappa_g^2(\ell)\kappa_n(\ell)\tau_g'(\ell) + \kappa_g(\ell)\kappa_n(\ell)\tau_g''(\ell) \end{aligned}$$

and

$$\begin{aligned} u_2(\ell) = \; & A(-2\kappa_g^2(\ell) + 2\kappa_n(\ell)\tau_g(\ell)) \\ & + 3B(\ell)C(\ell)\left(-\kappa_g'(\ell) + \kappa_g^2(\ell) - \kappa_n(\ell)\tau_g(\ell)\right) \\ & + B(\ell)(3\tau_g^2(\ell) - 3\kappa_g^2(\ell)\kappa_n(\ell)\tau_g(\ell) \\ & + 3\tau_g'(\ell)\kappa_n(\ell)\kappa_g). \end{aligned}$$

In order that $\left.\frac{dK}{dt}\right|_{t=0} = 0$ for all functions μ satisfying (19) with arbitrary values of $\mu(\ell)$ and $\mu'(\ell)$, the given arc γ must satisfy two boundary conditions;

$$u_1(\ell) = 0, \tag{3.11}$$

$$u_2(\ell) - u_1'(\ell) = 0 \tag{3.12}$$

and the differential equation;

$$\begin{aligned} & A(-\kappa_g\kappa_g' + \kappa_g^3 - 2\kappa_g\kappa_n\tau_g + \kappa_n\tau_g') + BC(-\kappa_g'' \\ & + 3\kappa_g\kappa_g' - 2\tau_g'\kappa_n - \kappa_g^3 + 2\kappa_g\kappa_n\tau_g) + B(3\tau_g\tau_g' - 2\tau_g^2\kappa_g \\ & - 2\kappa_g\kappa_g'\kappa_n\tau_g + \kappa_g^3\kappa_n\tau_g - \kappa_g^2\kappa_n\tau_g' + \kappa_g\kappa_n\tau_g'' - (A(-2\kappa_g^2 \\ & + 2\kappa_n\tau_g) + 3BC\left(-\kappa_g' + \kappa_g^2 - \kappa_n\tau_g\right) + B(3\tau_g^2 - 3\kappa_g^2\kappa_n\tau_g \\ & + 3\tau_g'\kappa_n\kappa_g))' + (A\kappa_g + BC(1 - 3\kappa_g) + 3B\kappa_g\kappa_n\tau_g)'' = 0 \end{aligned} \tag{3.13}$$

at the free end. Then we can give the following theorem.

Theorem 3.1. *The intrinsic equations for a null relaxed elastic curve of length ℓ having on a timelike surface S in Minkowski $3-$space $\mathbb{E}_1^3$ are given by the differential equation (3.13) together with the boundary conditions (3.11) and (3.12) .*

4. Examples

1. The most familiar instance of timelike surfaces in Minkowski $3-$space $\mathbb{E}_1^3$ is pseudo-sphere

$$S_1^2(1) = \left\{ p \in \mathbb{E}_1^3 \middle| <p, p>= 1 \right\}.$$

For all curves on $S_1^2(1)$ the geodesic torsion τ_g is zero and square of the normal curvature κ_n^2 is 1. Then, null geodesics of $S_1^2(r)$ are relaxed elastic lines.

2. The pseudo-cylinder

$$C_1^2(1) = \left\{ (x, y, z) \in \mathbb{E}_1^3 \middle| \; -x^2 + y^2 = 1, \; z \in \mathbb{R} \right\}$$

is a timelike surface and parametrized by $X(u, v) = (\sinh u, \cosh u, v)$. Then the null geodesic

$$\gamma(s) = (\sinh s, \cosh s, s)$$

with the geodesic torsion $\tau_g = -\frac{1}{2}$ and the normal curvature $\kappa_n = 1$ (see [1]) is a null relaxed elastic line.

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