SCREEN SEMI INVARIANT LIGHTLIKE SUBMANIFOLDS OF SEMI-RIEMANNIAN PRODUCT MANIFOLDS

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Abstract. In this paper, we introduce a new class of lightlike submanifold called screen semi-invariant (SSI) lightlike submanifolds of a semi-Riemannian product manifold. We give examples of such submanifolds and study the geometry of leaves of distributions which are involved in the definition of SSI-lightlike submanifolds. We obtain, necessary and sufficient conditions for the SSI-lightlike submanifold to be locally product manifold. Finally, we give some characterizations for totally umbilical SSI-lightlike and screen anti-invariant lightlike submanifolds of semi-Riemannian product manifolds.

1. Introduction

The geometry of lightlike submanifolds of semi-Riemannian manifolds is developed by K.L. Duggal-A.Bejancu [8] and K.L. Duggal and B. Şahin [4]. The lightlike submanifolds have been studied in various manifolds by many authors, [2], [3], [5], [6], [7]. In [3], K.L. Duggal and B. Şahin introduced a new class of lightlike submanifolds which is called Screen Cauchy Riemannian (SCR) lightlike submanifolds of indefinite Kaehler manifolds. They have shown that, SCR-lightlike submanifolds include invariant (complex) and screen real subcases of lightlike submanifolds. The geometry of submanifolds of a Riemannian product manifold (Semi-Riemannian Product manifold) have been extensively studied by many geometers, [12], [11],[10]. In case Riemannian, the invariant submanifolds and semi invariant submanifolds are investigated by Xinmin, L. and Shao, F.-M., [13]. As an analogue of CR-lightlike submanifolds, semi-invariant lightlike submanifolds were introduced by M. Atçeken and E. Kılıç [1]. Therefore, in [9], E.Kılıç and B. Şahin introduced radical anti-invariant lightlike submanifolds of semi-Riemannian product manifold. In this paper, we introduce a new class of lightlike submanifolds of semi-Riemannian product manifolds which is called screen semi invariant (SSI) lightlike manifold and investigate the geometry of such submanifolds.

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In Section 2 and Section 3, we give the basic concepts on lightlike submanifolds and product manifolds which will be used throughout this paper. In section 4, we introduce SSI-lightlike submanifolds and give examples. We investigate the integrability conditions of all the distributions. We also obtain that the SSI-lightlike submanifolds and its leave of the screen distribution are locally product manifolds under some conditions. In section 5, we study totally umbilical SSI-submanifolds and give a condition for its Ricci tensor to be symmetric. We prove that there exist no totally umbilical SSI-lightlike submanifolds in positively or negatively curved (or null sectional curved) semi-Riemannian product manifolds. Finally, in section 6, we study the geometry of screen anti-invariant lightlike submanifolds of semi-Riemannian Product manifolds.

2. Lightlike Submanifolds

In this paper, we use the same notations and terminologies as in [8].

Let \((\overline{M}, \overline{g})\) be an \((m+n)\)-dimensional semi-Riemannian manifold with index \(q > 0\) and \(M\) be a submanifold of \(n\)-codimension of \(\overline{M}\). If \(\overline{g}\) is degenerate on the tangent bundle \(T\overline{M}\) of \(\overline{M}\), then \(M\) is called a lightlike (degenerate) submanifold of \(\overline{M}\). We denote by \(\overline{g}\) the induced metric of \(\overline{g}\) on \(M\) and suppose that \(\overline{g}\) is degenerate, then for each tangent space \(T_xM\),

\[T_xM^\perp = \{U_x \in T_x\overline{M} : \overline{g}_x(U_x, V_x) = 0, \quad \forall V_x \in T_xM\},\]

is a degenerate \(n\)-dimensional subspace of \(T_x\overline{M}\). Thus both \(T_xM\) and \(T_xM^\perp\) are degenerate orthonormal distributions. In this case, there exists a subspace

\[\text{Rad}(T_xM) = T_xM \cap T_xM^\perp\]

which is called Radical subspace. The mapping

\[\text{Rad}(T_xM) : x \in M \longrightarrow \text{Rad}(T_xM)\]

defines a smooth distribution on \(M\) of rank(\(\text{Rad}(T_xM)\)) = \(r > 0\), then \(M\) is called \(r\)-lightlike submanifold and \(\text{Rad}(T_xM)\) is called radical distribution on \(M\).

There are four possible cases with respect to the dimension and codimension of \(M\) and rank of \(\text{Rad}(T_xM)\). We recall that

Case 1) \(M\) is called \(r\)-lightlike submanifold, if \(1 \leq r < \min\{m, n\}\).

Case 2) \(M\) is called co-isotropic submanifold, if \(1 \leq r = n < m\).

Case 3) \(M\) is called isotropic submanifold, if \(1 \leq r = m < n\).

Case 4) \(M\) is called totally lightlike submanifold, if \(1 \leq r = m = n\).

For Case 1, there exists a non-degenerate screen distribution \(S(TM)\) which is a complementary vector subbundle to \(\text{Rad}(TM)\) in \(TM\). Therefore, we can write

\[(2.1) \quad TM = \text{Rad}(TM) \perp S(TM),\]

As \(S(TM)\) is non-degenerate vector subbundle of \(T\overline{M} |_M\), we put

\[(2.2) \quad T\overline{M} |_M = S(TM) \perp S(TM)^\perp,\]
where $S(TM)^{±}$ is the complementary orthogonal vector subbundle of $S(TM)$ in $T\mathcal{M}|_M$. If we use the fact that $S(TM)$ and $S(TM)^{±}$ are non-degenerate, we have the following orthogonal direct decomposition

\begin{equation}
(2.3) \quad S(TM)^{±} = S(TM^{±}) \perp S(TM^{±})^⊥.
\end{equation}

Denote an $r$-lightlike submanifold by $(M, g, S(TM), S(TM^{±}))$.

**Theorem 2.1.** [8] Let $(M, g, S(TM), S(TM^{±}))$ be a $r$-lightlike submanifold of a semi-Riemannian manifold $(\mathcal{M}, \mathfrak{g})$. Then there exists a complementary vector bundle $\ell tr(TM)$ called a lightlike transversal bundle of $Rad(TM)$ in $S(TM^{±})^⊥$ and a basis of $\Gamma(\ell tr(TM)|_U)$ consists of smooth sections $\{N_1, ..., N_r\}$ of $S(TM^{±})^⊥ |_U$ such that

\begin{equation}
\mathfrak{g}(N_i, N_j) = \delta_{ij}, \quad \mathfrak{g}(N_i, N_j) = 0, \quad i, j = 1, ..., r,
\end{equation}

where $\{\xi_1, ..., \xi_r\}$ is a basis of $\Gamma(Rad(TM)|_U)$.

**Theorem 2.2.** [8] Let $M$ be an $r$-lightlike submanifold of a semi-Riemannian manifold $\mathcal{M}$. Then the induced connection $\nabla$ is a metric connection if and only if $Rad(TM)$ is a parallel distribution w.r.t. $\nabla$.

We consider the vector bundle

\begin{equation}
(2.4) \quad tr(TM) = \ell tr(TM) \perp S(TM^{±}).
\end{equation}

Thus we have

\begin{equation}
(2.5) \quad T\mathcal{M} = TM \oplus tr(TM) = S(TM) \perp S(TM^{±}) \perp (Rad(TM) \oplus \ell tr(TM)).
\end{equation}

Now, let $\nabla$ be the Levi-Civita connection on $\mathcal{M}$ and $\nabla$ be induced connection on $M$. Then the Gauss and Weingarten formulas are respectively given by

\begin{equation}
(2.6) \quad \nabla_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM)
\end{equation}

and

\begin{equation}
(2.7) \quad \nabla_X V = -A_Y X + \nabla_X^\perp V, \quad \forall X \in \Gamma(TM)
\end{equation}

for any $V \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_Y X\}$ and $\{h(X, Y), \nabla_X^\perp V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. It follows that $\nabla^\perp$ is linear connections on $tr(TM)$. Using the projections $L : tr(TM) \longrightarrow \ell tr(TM)$ and $S : tr(TM) \longrightarrow S(TM^{±})$, then we have

\begin{equation}
(2.8) \quad \nabla_X Y = \nabla_X^\perp Y + h^l(X, Y) + h^s(X, Y)
\end{equation}

\begin{equation}
(2.9) \quad \nabla_X N = -A_N X + \nabla_X^\perp N + D^s(X, N)
\end{equation}

and

\begin{equation}
(2.10) \quad \nabla_X W = -A_W X + \nabla_X^\perp W + D^f(X, W),
\end{equation}

for any $X, Y \in \Gamma(TM), N \in \Gamma(\ell tr(TM))$ and $W \in \Gamma(S(TM^{±}))$, where $h^l(X, Y) = L h(X, Y), h^s(X, Y) = Sh(X, Y), \nabla_X Y, A_N X, A_W X \in \Gamma(TM), \nabla_X^\perp N, D^f(X, W) \in \Gamma(\ell tr(TM))$ and $\nabla_X^\perp W, D^s(X, N) \in \Gamma(S(TM^{±}))$.

By using (2.8), (2.9) and (2.10) we obtain

\begin{equation}
(2.11) \quad \mathfrak{g}(h^s(X, Y), W) + \mathfrak{g}(Y, D^f(X, W)) = g(A_W X, Y).
\end{equation}
We denote the projection morphism of $TM$ to the screen distribution $S(TM)$ by $P$. According to (2.1) we have

\begin{equation}
\nabla_X P Y = \nabla_X^* P Y + h^*(X, P Y)
\end{equation}

(2.12)

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla_X^* P Y, A^*_X Y\}$ and $\{h^*(X, P Y), \nabla_X^* \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. It follows that $\nabla^*$ and $\nabla^*_\xi$ are linear connections on $S(TM)$ and $Rad(TM)$, respectively. Then we have the following equations

\begin{equation}
\bar{\sigma}(h^\ell(X, P Y), \xi) = g(A^*_X P X, P Y), \bar{\sigma}(h^* (X, P Y), N) = g(A_N X, P Y)
\end{equation}

(2.14)

\begin{equation}
g(A^*_X P X, P Y) = g(P X, A^*_X P Y), A^*_X \xi = 0
\end{equation}

(2.15)

for any $X, Y \in \Gamma(TM), \xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(tr(TM))$.

In general, the induced connection on lightlike submanifold $M$ is not metric connection. Since $\nabla$ is metric connection, $\nabla g$ is obtained from (2.6) and (2.8) as

\begin{equation}
(\nabla_X g)(Y, Z) = \bar{\sigma}(h^\ell(X, Y), Z) + \bar{\sigma}(h^*(X, Z), Y)
\end{equation}

(2.16)

for any $X, Y, Z \in \Gamma(TM)$. If $M$ is a real space form with constant sectional curvature $c$, then the Riemannian curvature tensor $\bar{R}$ of $\bar{M}$ is given by

\begin{equation}
\bar{R}(X, Y)Z = c\{\bar{\sigma}(Y, Z)X - \bar{\sigma}(X, Z)Y\},
\end{equation}

(2.17)

for any $X, Y, Z \in \Gamma(TM)$.

Now, we recall that the equation of Gauss for the lightlike immersion of $M$ in $\bar{M}$ is given by

\begin{equation}
\begin{aligned}
\bar{R}(X, Y)Z & = R(X, Y)Z + A_{h^\ell(X, Z)}Y - A_{h^*(Y, Z)}X + (\nabla_X h^\ell)(Y, Z) \\
& - (\nabla_Y h^\ell)(X, Z) + A_{h^*(X, Z)}Y + D^\ell(X, h^*(Y, Z)) \\
& - A_{h^*(Y, Z)}X - D^*(Y, h^*(X, Z)) + (\nabla_X h^*)(Y, Z) \\
& - (\nabla_Y h^*(X, Z) + D^*(X, h^\ell(Y, Z)) - D^*(Y, h^\ell(X, Z))
\end{aligned}
\end{equation}

(2.18)

for any $X, Y, Z \in \Gamma(TM)$.

We refer to [8] for the dependence of all the induced geometric objects of $M$ on $\{S(TM), S(TM^\perp)\}$.

3. **Semi-Riemannian Product Manifolds**

Let $(M_1, g_1)$ and $(M_2, g_2)$ be two $m_1$ and $m_2$-dimensional semi-Riemannian manifolds with constant indexes $q_1 > 0$, $q_2 > 0$, respectively. Let $\pi : M_1 \times M_2 \longrightarrow M_1$ and $\sigma : M_1 \times M_2 \longrightarrow M_2$ be the projections which are given by $\pi(x, y) = x$
and \( \sigma(x, y) = y \) for any \((x, y) \in M_1 \times M_2\). We denote the product manifold by \( M = (M_1 \times M_2, \bar{g}) \), where \[
\bar{g}(X, Y) = g_1(\pi_* X, \pi_* Y) + g_2(\sigma_*, \sigma_*)
\] for any \( X, Y \in \Gamma(TM) \) and \(*\) means tangent mapping. Then we have \( \pi_*^2 = \pi_* \), \( \sigma_*^2 = \sigma_* \), \( \pi_* \sigma_* = \sigma_* \pi_* = 0 \) and \( \pi_* + \sigma_* = I \), where \( I \) is identity transformation. Thus \((\bar{M}, \bar{g})\) is a \((m_1 + m_2)\)-dimensional semi-Riemannian manifold with constant index \((q_1 + q_2)\). The semi-Riemannian product manifold \( \bar{M} = M_1 \times M_2 \) is characterized by \( M_1 \) and \( M_2 \) which are totally geodesic submanifolds of \( \bar{M} \).

Now, if we put \( F = \pi_* - \sigma_* \), then we can easily see that \( F^2 = I \) and \[
\bar{g}(FX, Y) = \bar{g}(X, FY),
\] for any \( X, Y \in \Gamma(T\bar{M}) \). Then it can be seen that
\[
(\nabla_X F)Y = 0,
\] for any \( X, Y \in \Gamma(T\bar{M}) \), that is, \( F \) is parallel with respect to \( \nabla \) [12].

The Riemannian curvature tensor field of \( M_1 \times M_2 \) satisfied
\[
\bar{R}(X, Y)FZ = F\bar{R}(X, Y)Z,
\] for any \( X, Y, Z \in \Gamma(TM_1 \times TM_2) \).

Now, suppose that \( M_1 \) and \( M_2 \) are real space forms with constant sectional \( c_1 \) and \( c_2 \), respectively. Then the Riemannian curvature tensor \( \bar{R} \) of \( M = M_1(c_1) \times M_2(c_2) \) is given by
\[
\bar{R}(X, Y)Z = \frac{1}{16}(c_1 + c_2)\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(FY, Z)FX - \bar{g}(FX, Z)FY + \bar{g}(FY, Z)X - \bar{g}(FX, Z)Y + \bar{g}(Y, Z)FX - \bar{g}(X, Z)FY\}
\] for any \( X, Y, Z \in \Gamma(T\bar{M}) \) [14].

Let \( M \) be a submanifold of a Riemannian (or semi-Riemannian) product manifold \( \bar{M} = M_1 \times M_2 \). If \( F(TM) = TM \), then \( M \) is called invariant submanifold, if \( F(TM) \subset TM^+ \), then \( M \) is called anti-invariant submanifold.

4. SCREEN SEMI INVARIANT LIGHTLIKE SUBMANIFOLDS OF A PRODUCT MANIFOLD

In this section, we introduce \textit{Screen Semi-Invariant (SSI)} submanifolds of semi-Riemannian product manifolds, give examples and investigate the geometry of leaves of distributions.

**Definition 4.1.** Let \((\bar{M}, \bar{g})\) be a semi-Riemannian product manifold and \( M \) be a lightlike submanifold of \( \bar{M} \). We say that \( M \) is SSI-lightlike submanifold of \( \bar{M} \) if the following statements are satisfied:

1) There exists a non-null distribution \( D \subseteq S(TM) \) such that
\[
S(TM) = D \perp D^\perp, \quad FD = D, \quad FD^\perp \subseteq S(TM^+), \quad D \cap D^\perp = \{0\},
\]
where \( D^\perp \) is orthogonal complementary to \( D \) in \( S(TM) \).

2) \( \text{Rad } TM \) is invariant with respect to \( F \), that is \( F\text{Rad}(TM) = \text{Rad}(TM) \).
Then we have
\begin{align}
(4.2) \quad Fltr(TM) &= \ell tr(TM), \\
(4.3) \quad TM &= D' \perp D^\perp, \quad D' = D \perp \text{Rad}(TM).
\end{align}
Hence it follows that $D'$ is also invariant with respect to $F$. We denote the orthogonal complement to $FD^\perp$ in $S(TM^\perp)$ by $D_0$. Then, we have
\begin{align}
(4.4) \quad tr(TM) &= \ell tr(TM) \perp FD^\perp \perp D_0.
\end{align}
If $D \neq \{0\}$ and $D^\perp \neq \{0\}$, then we say that $M$ is a proper SSI-lightlike submanifold of $\overline{M}$. Hence, for on proper $M$, we have $\dim(D) \geq 1$, $\dim(D^\perp) \geq 1$, $\dim(M) \geq 3$ and $\dim(\overline{M}) \geq 5$. Furthermore, there exists no proper SSI-lightlike hypersurface of a semi-Riemannian product manifold.

If $D = \{0\}$, that is $FS(TM) \subseteq S(TM^\perp)$, then we say that $M$ is screen anti-invariant lightlike submanifold.

**Example 4.1.** Let $M_1$ and $M_2$ be $\mathbb{R}^3_1$ and $\mathbb{R}^2$, respectively. Then $\overline{M} = M_1 \times M_2$ is a semi-Riemannian product manifold with metric tensor $\overline{g} = \pi^* g_1 + \sigma^* g_2$, where $g_1$ and $g_2$ are the standard metric tensors of $\mathbb{R}^3_1$ and $\mathbb{R}^2$ with $(-, +, +)$ and $(+, +)$, $\pi_*$ and $\sigma_*$ are the projections of $\Gamma(\overline{M})$ to $\Gamma(TM_1)$ and $\Gamma(TM_2)$, respectively. Let $M$ be a submanifold of $\overline{M}$ given by equations
\begin{align*}
x^1 &= \sqrt{2} u_1 + u_3, \quad x^2 = u_1 + u_3, \quad x^3 = u_1 + (\sqrt{2} - 1)u_3, \\
x^4 &= u_2 + (\sqrt{2} - 1)u_3, \quad x^5 = u_2 - (\sqrt{2} - 1)u_3,
\end{align*}
where $u_1, u_2, u_3$ are real parameters. Then $TM$ is spanned by $\{U_1, U_2, U_3\}$, where
\begin{align*}
U_1 &= \sqrt{2} \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \quad U_2 = \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5}, \\
U_3 &= \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + (\sqrt{2} - 1) \frac{\partial}{\partial x^3} \quad \frac{\sqrt{2} - 1}{\sqrt{2}} \frac{\partial}{\partial x^4} - \frac{\sqrt{2} - 1}{\sqrt{2}} \frac{\partial}{\partial x^5}.
\end{align*}
Hence $M$ is a 1-lightlike submanifold with $Rad(TM) = \text{Span}(U_1)$. $S(TM)$ and $S(TM)^\perp$ are spanned by $\{U_2, U_3\}$ and $\{H\}$, respectively, where
\begin{align*}
H &= \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + (\sqrt{2} - 1) \frac{\partial}{\partial x^3} - \frac{\sqrt{2} - 1}{\sqrt{2}} \frac{\partial}{\partial x^4} + \frac{\sqrt{2} - 1}{\sqrt{2}} \frac{\partial}{\partial x^5}.
\end{align*}
Then the lightlike transversal vector bundle $ltr(TM)$ is spanned by
\begin{align*}
N &= -\frac{1}{2\sqrt{2}} \frac{\partial}{\partial x^1} + \frac{1}{4} \frac{\partial}{\partial x^2} + \frac{1}{4} \frac{\partial}{\partial x^3}.
\end{align*}
Therefore, $D = \text{Span}(U_2)$, $D^\perp = \text{Span}(U_3)$, $D_0 = \{0\}$ and $F Rad(TM) = Rad(TM)$, $FD = D$, $FD^\perp = S(TM^\perp)$, $Fltr(TM) = ltr(TM)$. Thus, $M$ is a proper SSI-lightlike submanifold of $\overline{M}$ with $D' = \text{Span}(U_1, U_2)$.

**Proposition 4.1.** Let $M$ be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\overline{M} = M_1 \times M_2$. Then $M$ is an invariant lightlike submanifold of $\overline{M}$ if and only if $D^\perp = \{0\}$.

**Proof.** If $M$ is a invariant lightlike submanifold of $\overline{M}$, then $FTM = TM$ and $D^\perp = \{0\}$. Conversely, if $D^\perp = \{0\}$, then $FTM = TM$. \qed
From this Proposition, we have the following Corollary.

**Corollary 4.1.** Let $M$ be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M} = M_1 \times M_2$. If $M$ is a co-isotropic or isotropic or totally lightlike, then $M$ is an invariant lightlike submanifold.

**Example 4.2.** Let $M_1$ and $M_2$ be $\mathbb{R}^2_1$ and $\mathbb{R}^2_2$ with standard metrics $g_1$ and $g_2$, respectively. Consider a submanifold $\mathcal{M}$ in $M_1 \times M_2$ given by the equations

$$x_3 = x_1 \cos \alpha - x_5 \sin \alpha, \quad x_4 = -x_1 \sin \alpha - x_5 \cos \alpha, \quad x_6 = \sqrt{2} x_5,$$

where $(x_1, x_2, x_3, x_4)$ and $(x_5, x_6)$ are standard coordinate systems of $\mathbb{R}^2_1$ and $\mathbb{R}^2_2$, respectively. Then $TM$ is spanned by

\[
Z_1 = \frac{\partial}{\partial x_1} + \cos \alpha \frac{\partial}{\partial x_3} - \sin \alpha \frac{\partial}{\partial x_4},
\]

\[
Z_2 = \frac{\partial}{\partial x_2},
\]

\[
Z_3 = -\sin \alpha \frac{\partial}{\partial x_3} - \cos \alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \sqrt{2} \frac{\partial}{\partial x_6}.
\]

Thus $M$ is a 1-lightlike submanifold with invariant $\text{Rad}(TM) = \text{Span}\{Z_1\}$. The screen distribution $S(TM) = \text{Span}\{Z_2, Z_3\}$ and $D = \text{Span}\{Z_2,\}; D^\perp = \text{Span}\{Z_3\}$.

On the other hand $S(TM^\perp)$ is spanned by $W_1 = \sin \alpha \frac{\partial}{\partial x_3} + \cos \alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \sqrt{2} \frac{\partial}{\partial x_6}$ and $W_2 = \sqrt{2} \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}$ and the lightlike transversal bundle $\ell tr(TM)$ is spanned by $N = -\frac{1}{2} \frac{\partial}{\partial x_1} + \frac{1}{2} \cos \alpha \frac{\partial}{\partial x_3} - \frac{1}{2} \sin \alpha \frac{\partial}{\partial x_4}$. Hence, $FD = D$, $FD^\perp \subset S(TM^\perp)$ and $M$ is a proper SSI-lightlike submanifold of $M_1 \times M_2$.

Let $M$ be a lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M} = M_1 \times M_2$. Then, for each $X \in \Gamma(TM)$ and $V \in \Gamma(\ell tr(TM))$, we put

\[
FX = fX + \omega X, \quad FV = BV + CV
\]

where $fX$, $BV$ and $\omega X$, $CV$ are the tangent and the transversal parts of $FX$ and $FV$. If $M$ is a SSI-lightlike submanifold of $\mathcal{M}$, then $fX \in \Gamma(D')$ and $\omega X \in \Gamma(FD^\perp)$, respectively.

**Theorem 4.1.** Let $M$ be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M} = M_1 \times M_2$. Then the screen distribution of $M$ is integrable if and only if the following three conditions are satisfied

\[
\text{(4.6)} \quad \overline{g}(AN Y, FX) = \overline{g}(AN X, FY), \quad X, Y \in \Gamma(D),
\]

\[
\text{(4.7)} \quad \overline{g}(AN Y, FX) = -\overline{g}(D^\alpha(X, N), FY), \quad X \in \Gamma(D), \quad Y \in \Gamma(D^\perp),
\]

\[
\text{(4.8)} \quad \overline{g}(D^\alpha(X, N), FY) = \overline{g}(D^\alpha(Y, N), FX), \quad X, Y \in \Gamma(D^\perp).
\]

**Theorem 4.2.** Let $M$ be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M} = M_1 \times M_2$. Then the distribution $D'$ is integrable if and only if $h(X, FY) = h(FX, Y)$, for all $X, Y \in \Gamma(D')$.

These last two theorems are similar to Theorem 3.3 and Theorem 3.4 given in [3], respectively.

**Theorem 4.3.** Let $M$ be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M} = M_1 \times M_2$. Then the distribution $D^\perp$ is integrable if and only if $A_{FZ}W = A_{FW}Z$, for any $W, Z \in \Gamma(D^\perp)$. 
Proof. Since \( F \) is parallel with respect to \( \nabla \), from (2.8), (2.10) and (4.5), we get
\[-A_{FW}Z + D'(Z, FW) + \nabla_Z FW = f\nabla_Z W + \omega\nabla_Z W + Bh(Z, W) + Ch(Z, W)\]
for all \( W, Z \in \Gamma(D^\perp) \). Taking tangential part of this equation, we have
\[-A_{FW}Z = f\nabla_Z W + Bh(Z, W).\]
(4.9)
By replacing role of vector fields \( W \) and \( Z \) in (4.9), by a direct calculation, we obtain
\[-A_{FZ}W + A_{FW}Z = f[Z, W].\]
Since \([Z, W] = f[Z, W] + \omega[Z, W]\), \( D^\perp \) is integrable if and only if \( f[Z, W] = 0 \) and we complete the proof.

Corollary 4.2. Let \( M \) be a SSI-lightlike submanifold of a semi-Riemannian product manifold \( \overline{M} = M_1 \times M_2 \). If the distribution \( D^\perp \) is integrable, then the following statements holds.

a) \( A_N \) is self-adjoint on \( D^\perp \) with respect to \( g \), for any \( N \in \Gamma(\text{tr}(TM)) \).

b) \( A_{FZ}W \) has no components in \( D^\perp \), for any \( Z, W \in \Gamma(D^\perp) \).

Proof. Suppose that \( D^\perp \) is integrable. Then, \( A_{FZ}W = A_{FW}Z \), for any \( Z, W \in \Gamma(D^\perp) \). Since \( \overline{g}(FW, FN) = \overline{g}(W, N) = 0 \) and \( \nabla \) is a metric connection, we obtain
\[\overline{g}(A_{FW}Z, FN) = -g(W, ANZ), \quad \overline{g}(A_{FZ}W, FN) = -g(Z, ANW),\]
for any \( N \in \Gamma(\text{tr}(TM)) \). From this last two equations, we have \( g(Z, ANW) = g(W, ANZ) \).

Since \( D^\perp \) is integrable, \( \overline{g}([Z, W], FX) = 0 \), for any \( Z, W \in \Gamma(D^\perp), X \in \Gamma(D) \). From (2.11), we have
\[\overline{g}(h^s(Z, X), FW) = g(A_{FW}Z, X).\]
(4.10)
Using (2.8) and (2.10), we obtain
\[\overline{g}(h^s(X, Z), FW) = -g(A_{FZ}X, W).\]
(4.11)
From (4.10) and (4.11), we have
\[g(A_{FW}Z, X) = -g(A_{FZ}X, W).\]
(4.12)
Since \( \nabla \) is a metric connection and \( \overline{g}(Z, FX) = 0 \) and using to symmetric of \( h^s \), we obtain
\[g(A_{FZ}W, X) = g(A_{FZ}X, W).\]
(4.13)
From (4.12) and (4.13), we have
\[g(A_{FW}Z, X) = -g(A_{FZ}W, X).\]
(4.14)
On the other hand, we get
\[\overline{g}([Z, W], FX) = g(A_{FZ}W, X) - g(A_{FW}Z, X) = 2g(A_{FZ}W, X) = 0.\]
Thus we have (b).
Theorem 4.4. Let $M$ be an SSI-lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M} = M_1 \times M_2$. Then the distribution $D$ is integrable if and only if the following statements hold:

a) $A_N$ is self-adjoint on $D$, for any $N \in \Gamma(\ell tr(TM))$.

b) $g(FY, A_U X) = g(FX, A_U Y)$, \( X, Y \in \Gamma(D) \) and $U \in \Gamma(FD^\perp)$.

Proof. Suppose that $D$ is integrable. Then, $[X, Y] \in \Gamma(D)$, that is $\overline{g}([X, Y], N) = 0$ and $\overline{g}([X, Y], FU) = 0$, \( X, Y \in \Gamma(D) \), $N \in \Gamma(\ell tr(TM))$ and $U \in \Gamma(FD^\perp)$. Thus we have

\begin{align*}
\overline{g}([X, Y], N) &= g(Y, A_N X) - g(X, A_N Y), \\
\overline{g}([X, Y], FU) &= g(FY, A_U X) - g(FX, A_U Y).
\end{align*}

Hence, from (4.15) and (4.16), we obtain (a) and (b), respectively.

Conversely, (a) and (b) are satisfied. From (4.15) and (4.16), we have $[X, Y] \in \Gamma(D)$, for any $X, Y \in \Gamma(D)$. \( \square \)

Theorem 4.5. Let $M$ be an SSI-lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M} = M_1 \times M_2$. Then the following assertions are equivalent:

1) The distribution $D$ is parallel.

2) $A_{FZ}$ is $S(TM)$-valued for $Z \in \Gamma(D^\perp)$.

3) $D^s(X, FN)$ is $D_0$-valued, for $X \in \Gamma(TM)$, $N \in \Gamma(\ell tr(TM))$.

Proof. $S(TM)$ is parallel if and only if $\overline{g}(\nabla_X Z, N) = 0$, for any $X, Z \in \Gamma(S(TM))$ and $N \in \Gamma(\ell tr(TM))$. Since $\overline{g}(\nabla_X Z, N) = \overline{g}(\nabla_X Z, N)$ and $F$ is parallel with respect to $\nabla$, we obtain

\begin{align*}
\overline{g}(\nabla_X Z, N) &= \overline{g}(\nabla_X FZ, FN).
\end{align*}

If $Z \in \Gamma(D^\perp)$, then $\overline{g}(A_{FZ} X, N) = 0$, that implies (b). Since $\nabla$ is a Levi-Civita connection, from (4.17), we get $\overline{g}(FZ, D^s(X, FN)) = 0$. Thus we have (c). \( \square \)

Theorem 4.6. Suppose that the screen distribution of $M$ is an SSI-lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M} = M_1 \times M_2$ is integrable. Then the following statements are equivalent:

1) The distribution $D$ defines a totally geodesic folation in $S(TM)$.

2) $Bh^s(X, Y) = 0$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.

3) $A_{FZ} X$ has no components in $D$, for any $X \in \Gamma(TM)$ and $Z \in \Gamma(D^\perp)$.

Proof. We assume that $D$ is totally geodesic in $S(TM)$. Then $\nabla_X^s Y \in \Gamma(D)$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Thus we have $g(\nabla_X F Y, Z) = 0$, for any $Z \in \Gamma(D^\perp)$. From (2.6) and (2.12), we get

\begin{align*}
g(\nabla_X^s FY, Z) &= g(\nabla_X Y, FZ) = 0.
\end{align*}

From (2.8), we have

\begin{align*}
\overline{g}(h^s(X, Y), FZ) &= 0.
\end{align*}

Hence we obtain (2). Since $\nabla$ is a Levi-Civita connection, we get

\begin{align*}
\overline{g}(\nabla_X Y, FZ) &= g(A_{FZ} X, Y) = 0.
\end{align*}

Thus we have (3). \( \square \)

It is easy check that, $D$ is totally geodesic in $S(TM)$ if and only if $D^\perp$ is totally geodesic in $S(TM)$. So we have following corollary.
Corollary 4.3. Suppose that the screen distribution of $M$ be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\mathbb{M} = M_1 \times M_2$ is integrable. Then $S(TM)$ is a locally product manifold if and only if $A_{FZ}X$ has no components in $D$, for any $X \in \Gamma(TM)$ and $Z \in \Gamma(D^\perp)$.

Theorem 4.7. Let $M$ be a SSI-lightlike submanifold of a semi-Riemannian product manifold $\mathbb{M} = M_1 \times M_2$. Then $M$ is a locally product manifold if and only if $\nabla f = 0$

Proof. Let $M$ be a locally product manifold. Then the leaves of distributions $D'$ and $D^\perp$ are both totally geodesic in $M$. Since $\nabla F = 0$ and from (2.6) and (2.7) we get

\[(4.18) \quad \nabla_X fY + h(X, fY) = f\nabla_X Y + \omega \nabla_X Y + Bh(X, Y) + Ch(X, Y),\]

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D')$. Since $D'$ is totally geodesic in $M$, $\nabla_X Y \in \Gamma(D)$. Then, for any $U \in \Gamma(FD^\perp)$, we have

\[\nabla_X(U) = g(\nabla_X Y, FU) = 0.\]

Hence we get $Bh(X, Y) = 0$. Comparing the tangential and transversal parts with respect to $D$ of equation (4.18), $\nabla_X fY = f\nabla_X Y$, that is $(\nabla_X f)Y = 0$.

Similarly,

\[(4.19) \quad -A_{FZ}X + \nabla_X FZ = f\nabla_X Z + \omega \nabla_X Z + Bh(X, Z),\]

for any $X \in \Gamma(TM)$ and $Z \in \Gamma(D^\perp)$. From (4.19), we have

\[-A_{FZ}X = f\nabla_X Z + Bh(X, Z).\]

For any $Y \in \Gamma(D')$, we get

\[g(\nabla_X Z, Y) = -g(A_{FZ}X, Y) = -g(\nabla_X fY, Z) = 0,\]

that is $f\nabla_X Z = 0$, which implies that $(\nabla_X f)Z = 0$.

Conversely, we suppose that $\nabla f = 0$. Then we have $\nabla_X fY = f\nabla_X Y$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D')$. Thus $\nabla_X fY \in \Gamma(D)$ and the distribution $D'$ is totally geodesic in $M$. Similarly, $\nabla_X fZ = f\nabla_X Z = 0$, for any $X \in \Gamma(TM)$, $Z \in \Gamma(D^\perp)$ and $D^\perp$ is totally geodesic in $M$. □

5. Totally Umbilical SSI-Lightlike Submanifolds

In this section, we study totally umbilical SSI-Lightlike submanifolds of a semi-Riemannian product manifold.

Definition 5.1. [7] A lightlike submanifold $(M, g)$ of a semi-Riemannian manifold $(\mathbb{M}, g)$ is called totally umbilical in $\mathbb{M}$, if there is a smooth transversal vector field $\mathcal{H} \in \Gamma(tr(TM))$ on $M$, called the transversal curvature vector field of $M$, such that, for all $X, Y \in \Gamma(TM)$,

\[h(X, Y) = g(X, Y)\mathcal{H}.\]

It is known that $M$ is totally umbilical if and only if on each coordinate neighborhood $U$, there exist smooth vector fields $\mathcal{H}^t \in \Gamma(tr(TM))$ and $\mathcal{H}^s \in \Gamma(S(TM^\perp))$ such that

\[h^t(X, Y) = g(X, Y)\mathcal{H}^t, \quad h^s(X, Y) = g(X, Y)\mathcal{H}^s,\]

for any $X, Y \in \Gamma(TM)$. 

Corollary 5.1. Let $M$ be a totally umbilical SSI-lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M} = M_1 \times M_2$. Then the distribution $D^\perp$ is totally geodesic in $M$.

Proof. Let $X, Y \in \Gamma(D^\perp)$. Then we have
$$\nabla_X Y = \tilde{\nabla}_X Y + \tilde{h}(X, Y),$$
where $\tilde{\nabla}_X Y \in \Gamma(D^\perp)$ and $\tilde{h}(X, Y) \in \Gamma(D')$. Since $D'$ is a invariant distribution, for any $Z \in \Gamma(D')$, we have $FZ = fZ \in \Gamma(D')$. Since $\tilde{\nabla}$ is a Levi-Civita connection, it can be easily calculated
$$g(\tilde{h}(X, Y), FZ) = g(\nabla_X Y, FZ) = g(\nabla_X Y, FZ) = -g(FY, h^s(X, Z)).$$
Since $X \in \Gamma(D^\perp)$ and $Z \in \Gamma(D')$, from (5.2), we have $h^s(X, Z) = 0$, and we have assertion of corollary.

Theorem 5.1. Let $M$ be a totally umbilical SSI-lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M} = M_1 \times M_2$. Then the following assertions are equivalent:
1) The distribution $D'$ is totally geodesic in $M$.
2) $A_{FZ}$ is $D^\perp$-valued, for any $Z \in \Gamma(D^\perp)$.
3) $H^s \in \Gamma(D_0)$.

Proof. Let $X, Y \in \Gamma(D')$. Then we have
$$\nabla_X Y = \tilde{\nabla}_X Y + h^t(X, Y),$$
where $\tilde{\nabla}_X Y \in \Gamma(D')$ and $h^t(X, Y) \in \Gamma(D^\perp)$. Since $\tilde{\nabla}$ is a Levi-Civita connection, it can be easily calculated
$$g(h^t(X, FY), Z) = g(h^s(X, Y), Z) = g(FY, A_{FZ} X),$$
for any $Z \in \Gamma(D^\perp)$. Thus we have (1)-(3).

From Corollary 5.1 and Theorem 5.1, we have the following theorem.

Theorem 5.2. Let $M$ be a totally umbilical SSI-lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M} = M_1 \times M_2$. Then $M$ is a locally product manifold if and only if $H^s \in \Gamma(D_0)$.

Theorem 5.3. Let $M$ be a totally umbilical SSI-lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M} = M_1(e_1) \times M_2(e_2)$. Then, the Ricci tensor on $M$ is symmetric if and only if $A_{H^t}$ is self adjoint on $M$.

Proof. The Ricci tensor of a lightlike submanifold is given by
$$Ric(X, Y) = \sum_{i=1}^{m} \varepsilon_i g(R(e_i, X)Y, e_i) + \sum_{j=1}^{r} g(R(\xi_j, X)Y, N_j),$$
for any $X, Y \in \Gamma(TM)$, where $\{e_1, \ldots, e_m\}$ is a orthonormal basis of $\Gamma(S(TM))$, $\{\xi_1, \ldots, \xi_r\}$ and $\{N_1, \ldots, N_r\}$ are lightlike basis of $\Gamma(Rad TM)$ and $\Gamma(\ell tr(TM))$. 
Suppose that $M$, respectively and $\overline{\mathcal{F}}(N, \epsilon_j) = \delta_j$, for any $i, j \in \{1, ..., r\}$. From (3.2) and (18), we obtain

$$Ric(X, Y) - Ric(Y, X) = -g(A_{\mathcal{H}^\epsilon}X, Y) + g(A_{\mathcal{H}^\epsilon}Y, X)$$

$$- g(A_{\mathcal{H}^\epsilon}X, Y) + g(A_{\mathcal{H}^\epsilon}Y, X).$$

Suppose that Ricci tensor is symmetric on $M$. If $X, Y \in \Gamma(\text{Rad} TM)$, then we have $g(A_{\mathcal{H}^\epsilon}X, Y) = g(A_{\mathcal{H}^\epsilon}Y, X) = g(A_{\mathcal{H}^\epsilon}Y, X) = g(A_{\mathcal{H}^\epsilon}X, Y) = 0$.

If $X \in \Gamma(\text{Rad} TM)$ and $Y \in \Gamma(S(TM))$, from (2.11) we have $g(A_{\mathcal{H}^\epsilon}X, Y) = g(A_{\mathcal{H}^\epsilon}Y, X) = 0$.

If $X, Y \in \Gamma(S(TM))$, then from (2.11), we get $g(A_{\mathcal{H}^\epsilon}X, Y) = g(X, Y)g(\mathcal{H}^\epsilon, \mathcal{H}^\epsilon)$.

that is $-g(A_{\mathcal{H}^\epsilon}X, Y) + g(A_{\mathcal{H}^\epsilon}Y, X) = 0$. Thus we have our assertion. □

**Theorem 5.4.** There exist no totally umbilical proper SSI-lightlike submanifold with $\text{dim}(D) \geq 2$ in any negatively or positively curved (and also null sectional curved) semi-Riemannian product manifold.

**Proof.** Suppose that $M$ is totally umbilical proper SSI-lightlike submanifold in semi-Riemannian product manifold $M(c)$ with $c \neq 0$. From (2.19), for $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$, we have

$$\overline{\mathcal{F}}(\overline{\mathcal{R}}(X, Y)X, Y) = \overline{\mathcal{F}}(\overline{\mathcal{R}}(X, Y)FX, FY)$$

$$= \overline{\mathcal{F}}((\nabla_X h^\epsilon)(Y, FX), FY) - \overline{\mathcal{F}}((\nabla_Y h^\epsilon)(X, FX), FY).$$

From (5.2), we get

$$(\nabla_X h^\epsilon)(Y, FX) = -(g(\nabla_X Y, FX) + g(Y, \nabla_X FX))\mathcal{H}^\epsilon.$$

Since $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$, we have $\overline{\mathcal{F}}(FX, Y) = 0$. Since $\overline{\mathcal{F}}$ is parallel with respect to $\nabla$, we get

$$0 = X\overline{\mathcal{F}}(Y, FX) = g(\nabla_X Y, FX) + g(Y, \nabla_X FX).$$

Since $\text{dim}(D) \geq 2$, we chose $X \in \Gamma(D)$ such that $g(X, FX) = 0$. From (5.2), we obtain

$$(\nabla_Y h^\epsilon)(X, FX) = -2g(\nabla_Y X, FX)\mathcal{H}^\epsilon.$$

Therefore,

$$0 = Y\overline{\mathcal{F}}(X, FX) = 2g(\nabla_Y X, FX).$$

Hence, $\overline{\mathcal{F}}(\overline{\mathcal{R}}(X, Y)X, Y) = 0$ which is a contradiction. Similarly, it can be proved for the null sectional curved case. □

### 6. Screen Anti-Invariant Lightlike Submanifolds

In this section, we will investigate the screen anti-invariant lightlike submanifolds of semi-Riemannian product manifolds.

Let $M$ be a screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $(\overline{M}, \overline{g})$. Then we have

$$S(TM) = FS(TM) \perp D_0.$$

We say that $M$ is a proper screen anti-invariant lightlike submanifold, if $S(TM) \neq \{0\}$ and $D_0 \neq \{0\}$. Thus we have the following proposition.
Proposition 6.1. There exist no proper screen anti-invariant co-isotropic, isotropic or totally lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M}$.

Example 6.1. Consider in $\mathbb{R}^4_1 \times \mathbb{R}^4_1$ the submanifold $M$ given by
\[ x^1 = u_1 + u_2, x^2 = u_1 + u_2, x^3 = u_3, \quad y^1 = u_1 - u_2, y^2 = u_1 - u_2, y^3 = u_3, y^4 = 0, \]
where $(x^1, x^2, x^3)$ and $(y^1, y^2, y^3, y^4)$ are standard coordinate systems of $\mathbb{R}^4_1$, respectively, and $\mathbb{R}^4_1$ and $u_1, u_2, u_3$ are real parameters. Then we have
\[ TM = \text{Span}\{U_1 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^2}, \quad U_2 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^2}, \quad U_3 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial y^3}\}. \]
The radical distribution $\text{Rad}(TM)$ is spanned by $\{U_1, U_2\}$ and the screen distribution $S(TM)$ is spanned by $U_3$. Hence $M$ is a 2-lightlike submanifold of $\mathbb{R}^4_1 \times \mathbb{R}^4_1$.

Take
\[ S(TM^\perp) = \{V_1 = \frac{\partial}{\partial x^3} - \frac{\partial}{\partial y^3}, \quad V_2 = \frac{\partial}{\partial y^4}\}, \]
and by the direct calculations we get
\[ \ell tr(TM) = \text{Span}\{N_1 = -\frac{1}{2}\{2 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + 2 \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}\}, \quad N_2 = -\frac{1}{2}\{2 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} - 2 \frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^2}\}\}. \]
We easily check that, $\text{Rad}(TM)$ and $\ell tr(TM)$ are invariant distributions with respect to $F$ and $FS(TM) \subset S(TM^\perp)$, where $D_0 = \text{Span}\{V_1\}$. Thus $M$ is a screen anti-invariant lightlike submanifold.

Let $M$ be a screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M}$. Then, for any $X \in \Gamma(TM)$, we can write
\[ FX = \mathcal{F}X + \pi X, \]
where $\mathcal{F}X \in \Gamma(\text{Rad}(TM))$ and $\pi X \in \Gamma(FS(TM))$. Similarly, for any $V \in \Gamma(\ell tr(TM))$, we can write
\[ FV = \mathcal{B}V + \mathcal{C}V, \]
where $\mathcal{B}V \in \Gamma(S(TM))$ and $\mathcal{C}V \in \Gamma(\ell tr(TM))$.

Theorem 6.1. Let $M$ be a proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\mathcal{M}$. Then the induced connection $\nabla$ is a metric connection if and only if $h^s(X, \xi') \in \Gamma(D_0)$, for any $\xi' \in \Gamma(\text{Rad} TM), X \in \Gamma(TM)$.

Proof. If $\xi \in \Gamma(\text{Rad} TM)$, then there exists a $\xi' \in \Gamma(\text{Rad} TM)$ such that $\xi = F\xi'$. From (3.1) and Gauss formula, we get
\[ \nabla_X \xi + h(X, \xi) = \mathcal{F}X + \mathcal{W}X \xi' + \mathcal{B}h^s(X, \xi') + \mathcal{C}h^s(X, \xi'), \]
for any $X \in \Gamma(TM)$. If we take tangential component of this equation, we have
\[ \nabla_X \xi = \mathcal{F}X + \mathcal{B}h^s(X, \xi'). \]
Thus, the radical distribution $\text{Rad}(TM)$ is a parallel distribution if and only if $h^s(X, \xi') \in \Gamma(D_0)$. From Theorem 2.2 we have the assertion of the theorem. \qed
Theorem 6.2. Let $M$ be a proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the following assertion equivalent:
1) $S(TM)$ is integrable.
2) For any $X, Y \in \Gamma(S(TM))$, $N \in \Gamma(\ell tr(TM))$, $\overline{\eta}(A_{FY}X, N) = \eta(A_{FX}Y, N)$.
3) $\overline{\eta}(FY, D^s(X, N)) = \eta(FX, D^s(Y, N))$.

Proof. Suppose that $S(TM)$ is integrable. Then we have $\overline{\eta}([X, Y], FN) = 0$, for any $X, Y \in \Gamma(S(TM))$, $N \in \Gamma(\ell tr(TM))$. From (2.6) and (3.1) we have $\overline{\eta}(A_{FY}X, N) = \eta(A_{FX}Y, N)$. Since $\nabla$ is a metric connection, we get $\overline{\eta}(A_{FY}X, N) = \eta(FY, D^s(X, N))$ and (3) is satisfied. Since $\overline{\eta}([X, Y], FN) = \eta(FY, D^s(X, N)) - \eta(FX, D^s(Y, N))$, (3)$\Rightarrow$(1). \qed

Theorem 6.3. Let $M$ be a proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the radical distribution integrable if and only if
$$h^s(\xi, F\xi') = h^s(F\xi, \xi'),$$
for any $\xi, \xi' \in \Gamma(\text{Rad}(TM))$.

Proof. For any $\xi, \xi' \in \Gamma(\text{Rad}(TM))$ and $U \in \Gamma(\text{FS}(TM))$, from (2.8) and (3.1), we get
$$\overline{\eta}([\xi, \xi'], FU) = \eta(h^s(\xi, F\xi') - h^s(F\xi, \xi'), U).$$
This the assertion of the theorem. \qed

Theorem 6.4. Let $M$ be a proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the following assertion equivalent:
1) The screen distribution $S(TM)$ defines a totally geodesic foliation in $M$.
2) $A_{FY}$ is valued $S(TM)$, for all $Y \in \Gamma(S(TM))$.
3) For any $X \in \Gamma(TM)$ and $N \in \Gamma(\ell tr(TM))$, $D^s(X, N) \in \Gamma(D_0)$.

Proof. Suppose that $S(TM)$ is totally geodesic. Then, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(S(TM))$, $\nabla_X Y \in \Gamma(S(TM))$. Thus we have $\overline{\eta}(\nabla_X Y, FN) = \eta(\nabla_X FY, N) = 0$ and (2) is satisfied. Since $\nabla$ is a metric connection, we get $\overline{\eta}(\nabla_X FY, N) = -\eta(FY, D^s(X, N)) = 0$ and $D^s(X, N) \in \Gamma(D_0)$. This is complete of proof. \qed

Theorem 6.5. Let $M$ be a proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the screen distribution is a parallel distribution in $M$ if and only if $A_{Z_{FY}}$ is $S(TM)$ valued.

Proof. $S(TM)$ is parallel if and only if $\overline{\eta}(\nabla_X Y, FN) = 0$, for any $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(\ell tr(TM))$. Since $\overline{\eta}(\nabla_X Y, FN) = \eta(\nabla_X FY, N)$, we obtain $\overline{\eta}(\nabla_X Y, FN) = -\eta(A_{Z_{FY}}Y, N)$. Thus we have the assertion of the theorem. \qed

Theorem 6.6. Let $M$ be a proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then $M$ is a locally product manifold if and only if $\overline{\mathcal{T}}$ is parallel with respect to induced connection $\nabla$, that is, $\nabla \overline{\mathcal{T}} = 0$.

Proof. We suppose that $M$ is a locally product manifold. Then the leaves of the distributions of $\text{Rad}(TM)$ and $S(TM)$ are totally geodesic in $M$. Thus $\nabla_{\overline{\mathcal{T}}} = 0$. \qed
$\Gamma(\text{Rad}(TM))$, for any $Z \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad}(TM))$. Since $F\xi = \overline{\text{F}}\xi$, from (3.1) we get

$$0 = \nabla_Z F\xi - F(\nabla_Z \xi)$$

$$= \nabla_Z \overline{\text{F}}\xi - \overline{\text{F}}(\nabla_Z \xi) + h(Z, \overline{\text{F}}\xi) - Fh(Z, \xi).$$

If we take tangential component of this equation, we get $(\nabla_Z \overline{\text{F}})\xi = 0$. For any $X \in \Gamma(S(TM))$, $\nabla_Z \overline{\text{F}}X = 0$ and $\overline{\text{F}}(\nabla_Z X) = 0$. Thus we have $\overline{\text{F}}(\nabla_Z X) = 0$ and $\overline{\text{F}}$ is parallel.

Now suppose that $\overline{\text{F}}$ is parallel with respect to $\nabla$. Then

$$\nabla_Z \overline{\text{F}}X = \overline{\text{F}}(\nabla_Z X)$$

for any $X, Z \in \Gamma(TM)$. If $X \in \Gamma(\text{Rad}(TM))$, then we have $\nabla_Z \overline{\text{F}}X \in \Gamma(\text{Rad}(TM))$ and $\overline{\text{F}}(\text{Rad}(TM))$ is totally geodesic in $M$. If $X \in \Gamma(S(TM))$, then we have $\overline{\text{F}}X = 0$ and $\overline{\text{F}}(\nabla_Z X) = 0$, that is $\nabla_Z X \in \Gamma(S(TM))$.

Now, let $M$ be totally umbilical proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then, from (2.6) and (5.1) we have

$$\nabla_X \xi = \nabla_X \xi,$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad}(TM))$. Since $\text{Rad}(TM)$ is invariant distribution w.r.t. $F$, there exists a $\xi' \in \Gamma(\text{Rad}(TM))$ such that $\xi = F\xi'$. From above equation and (3.1), we get

$$(6.3) \quad \nabla_X \xi = \overline{\text{F}} \nabla_X \xi'.$$

Since $\overline{\text{F}} \nabla_X \xi' \in \Gamma(\text{Rad}(TM))$, then $\nabla_X \xi \in \Gamma(\text{Rad}(TM))$, i.e. the radical distribution is a parallel distribution in $M$. From Theorem 2.2, we have following corollary.

**Corollary 6.1.** Let $M$ be totally umbilical screen proper anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the induced connection $\nabla$ is a metric connection.

**Corollary 6.2.** Let $M$ be totally umbilical proper screen anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\overline{M}$. Then the radical distribution defines a totally geodesic foliation in $M$.

**Proof.** The radical distribution defines a totally geodesic foliation if and only if $\nabla_{\xi_1} \xi \in \Gamma(\text{Rad}(TM))$, for any $\xi_1, \xi \in \Gamma(\text{Rad}(TM))$. If we take $\xi_1$ for $X$ in equation (6.3), then we have $\nabla_{\xi_1} \xi \in \Gamma(\text{Rad}(TM))$. $\square$

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